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PREFACE.

THIS Treatise is intended to be a continuation of Mr. Hamblin Smith's *Algebra* (Part I.).

I have prefixed to it an Appendix to Part I., containing several propositions and proofs, which properly belong to the portion of Algebra treated of in that Part, but were thought too difficult for the student when first beginning the subject.

My thanks are due to several friends who have aided me with their suggestions; but I am especially indebted to Mr. J. H. Davis of Painswick Grammar School, and to Mr. G. R. Jellicoe of London. The former corrected the proof-sheets and worked through most of the examples, except in the portion relating to Probabilities, which the latter kindly undertook.

I wish also to acknowledge the help I have obtained in Probabilities from Mr. Venn's "Logic of Chance," as I

have attempted to make the view taken in that work the basis of my own explanation.

I cannot but fear that in a volume containing so many indices and suffixes several errors still remain undetected. I shall very thankfully receive any corrections, or suggestions for the improvement of the book, which may be sent to me.

E. J. GROSS.

GONVILLE AND CAIUS COLLEGE,
October, 1874.

CONTENTS.

APPENDIX TO PART I.

1. The following is an extension of the method of [Art. 215], to simultaneous equations involving *three* unknowns.

To solve the equations,

$$a_1x+b_1y+c_1z=d_1 \quad . \quad . \quad . \quad (1),$$
$$a_2x+b_2y+c_2z=d_2 \quad . \quad . \quad . \quad (2),$$
$$a_3x+b_3y+c_3z=d_3 \quad . \quad . \quad . \quad (3).$$

If we multiply equation (1) by $b_2c_3-b_3c_2$, (2) by $b_3c_1-b_1c_3$, (3) by $b_1c_2-b_2c_1$ and add, we obtain the equation,

$$x\{a_1(b_2c_3-b_3c_2)+a_2(b_3c_1-b_1c_3)+a_3(b_1c_2-b_2c_1)\}$$
$$=d_1(b_2c_3-b_3c_2)+d_2(b_3c_1-b_1c_3)+d_3(b_1c_2-b_2c_1) \quad . \quad (4).$$

Thus x is determined, and in a similar manner by multiplying the equations (1), (2), (3), by appropriate multipliers, we might obtain equations for separately determining y and z.

2. This is called the Method of Cross Multiplication.

It will be observed that to form any particular multiplier, such as that for equation (2) in finding x, we start with b_3 the co-efficient of the succeeding unknown in the succeeding equation, and multiply it by c_1 the co-efficient of the remaining unknown in the remaining equation, and from this subtract the product b_1c_3. In this observation we consider (1) as the equation succeeding to (3).

3. The expression for y obtained as above has the denominator

$$=b_1(c_2a_3-c_3a_2)+b_2(c_3a_1-c_1a_3)+b_3(c_1a_2-c_2a_1),$$

which the student will readily see is the same as the denominator in the expression for x obtained from (4).

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4. To eliminate x, y, z from the equations,

$$a_1x+b_1y+c_1z=0 \quad . \quad . \quad . \quad (1),$$
$$a_2x+b_2y+c_2z=0 \quad . \quad . \quad . \quad (2),$$
$$a_3x+b_3y+c_3z=0 \quad . \quad . \quad . \quad (3).$$

Multiply (1), (2), (3), respectively by $b_2c_3-b_3c_2$, etc., as in Art. 1, and we obtain

$$x\{a_1(b_2c_3-b_3c_2)+a_2(b_3c_1-b_1c_3)+a_3(b_1c_2-b_2c_1)\}=0,$$

or $$a_1(b_2c_3-b_3c_2)+\text{etc.},\qquad =0.$$

Note.—Any equation of the first degree is said to be linear.

EXAMPLES.—I.

Solve the equations

1. $a^2x+ay+z=-a^3$,
$b^2x+by+z=-b^3$,
$c^2x+cy+z=-c^3$.

2. $3x+2y+4z=19$,
$2x+5y+3z=21$,
$3x-y+z=4$.

3. $3x-4y+5z=9$,
$7x+2y-10z=18$,
$5x-6y-15z=6$.

4. $x-y+z=0$,
$(b+c)x-(c+a)y+(a+b)z=0$,
$bcx-cay+abz=1$.

5. $x+2y+3z=14$,
$2x-3y+4z=8$,
$3x+4y-3z=-4$.

6. $bz+cy=a$,
$cx+az=b$,
$ay+bx=c$.

7. $x+y+z=a+b+c$,
$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$,
$\frac{x}{a^3}+\frac{y}{b^3}+\frac{z}{c^3}=0$.

8. $lx+my+nz=\left(\sqrt{\frac{l}{m}}-\sqrt{\frac{m}{l}}\right)\left(\sqrt{\frac{m}{n}}-\sqrt{\frac{n}{m}}\right)\left(\sqrt{\frac{n}{l}}-\sqrt{\frac{l}{n}}\right)$,
$\frac{x}{l}+\frac{y}{m}+\frac{z}{n}=0$,
$x+y+z=0$.

5. We now return to the principle mentioned in [Art. 333]. But before we proceed to the proof of it, it will be useful to give a few explanations of the words and notation employed.

6. Any expression containing, or *involving*, a particular symbol, as x, is called a function of that symbol [Art. 331], and is often denoted by writing the symbol within brackets and some letter before the first bracket, thus $F(x)$, $f(x)$, $\phi(x)$ denote different functions of x.

7. $F(x)$, $F(a)$ denote the *same* functions of x and of a; for example, if $F(x)=x^2+2x-x^{-1}+6x^{\frac{2}{3}}+7$,

then $F(a)=a^2+2a-a^{-1}+6a^{\frac{2}{3}}+7$,

and $F(2)=2^2+2.2-2^{-1}+6.2^{\frac{2}{3}}+7$.

8. If we divide any *positive integral* function of x [Art. 331], such as $3x^4+5x^2-2x+1$, by a simple expression, such as $x-2$, it will be found that at each step the remainder is a positive integral function of x, but the highest power of x in it is lower by *one* than that in the preceding remainder, until at last we obtain a remainder not involving x at all. This, unless anything is expressly said to the contrary, is called *the* remainder when the given function is divided by the simple expression.

The following Proposition shows that we can obtain this remainder at once, by writing 2 for x in the dividend, or in other words, that the remainder is the same function of 2 that the dividend is of x.

9. PROP. *If* $\phi(x)$ *be any positive integral function of* x *and* $\phi(x)$ *be divided by* $x-a$, *then the remainder will be* $\phi(a)$, i.e. *it will be the same function of* a *as the dividend is of* x.

Suppose the division performed, and that Q is the quotient, and R the remainder;

$$\therefore \phi(x)=(x-a)Q+R \qquad . \quad . \qquad (1).$$

Now since $\phi(x)$ is a *positive integral* function of x, R cannot contain x, and therefore will remain unchanged whatever value we give to x; put then $x=a$ and we obtain from (1)

$$\phi(a)=R. \qquad \text{Q.E.D.}$$

COR. 1. If a be a root of the corresponding [Art. 332] equation $\phi(x)=0$, or in other words, if $\phi(a)=0$, then $R=0$, and the expression $\phi(x)$ is divisible by $x-a$.

COR. 2. *Conversely*, if the expression $\phi(x)$ be divisible by $x-a$, we have $R=0$, *i.e.* $\phi(a)=0$, and $\therefore$ a is a root of the corresponding equation $\phi(x)=0$.

From Cor. 1 we can often see whether a given expression is divisible by $x-a$; for if it be, it must vanish when in it we put $x=a$. [Art. 333, Ex. (2.)].

10. *Ex.* To show that xyz is a factor of the expression

$$(-x+y+z)(x-y+z)(x+y-z)+x(x-y+z)(x+y-z)$$
$$+y(x+y-z)(-x+y+z)+z(-x+y+z)(x-y+z).$$

In it put $x=0$, it becomes

$$(y+z)(-y+z)(y-z)+y(y-z)(y+z)+z(y+z)(-y+z)$$
$$=(y+z)(y-z)\{-y+z+y-z\}=0\,;\ \therefore x \text{ is a factor.}$$

By symmetry, y and z are also factors; $\therefore$ xyz is a factor.

11. *Obs.* From this example we can show that the given expression $=4xyz$.

For by the example xyz is a factor of it; $\therefore$ we can put

$$(-x+y+z)(x-y+z)(x+y-z)+x(x-y+z)(x+y-z)+\text{etc.}=Nxyz\,;$$

and if we were to multiply out the factors on the left-hand side, all the terms containing x^3, x^2y, etc. must cancel out, and leave only the terms containing xyz as a factor; and the numerical co-efficients of these terms, with their proper signs, must make up N, *i.e.* N is a number, and does not involve x, y, or z, and $\therefore$ remains unaltered, whatever value we give to x, y, and z. Put then $x=2$, $y=1$, $z=1$;

$$\therefore\ -x+y+z=0,\ x-y+z=2,\ x+y-z=2\,;$$
$$\therefore\ 2.2.2=N.2.1.1\,; \qquad \therefore\ N=4.$$

EXAMPLES.—II.

1. Write down the remainder after the division of $4x^3-3x^2-7$ by $x-3$.

2. Is $2x^4-5x^2+7x+1$ divisible by $x-1$?

3. Is $3x^5-7x^3+2x^2-40$ divisible by $x-2$?

4. Prove that

$$(a+b+c)(bc+ca+ab)-(b+c)(c+a)(a+b)=abc.$$

5. Prove that $(b-c)(x-b)(x-c)+(c-a)(x-c)(x-a)+(a-b)(x-a)(x-b)=-(b-c)(c-a)(a-b)$.

6. Prove that $BC(B-C)+CA(C-A)+AB(A-B)=-(B-C)(C-A)(A-B)$.

7. If $s=a+b+c$, prove that $s(s-2b)(s-2c)+s(s-2c)(s-2a)+s(s-2a)(s-2b)-(s-2a)(s-2b)(s-2c)=8abc$.

8. Prove that $(a+b+c)^4-(b+c)^4-(c+a)^4-(a+b)^4+a^4+b^4+c^4=12abc(a+b+c)$.

9. Prove that

$$\begin{aligned}&(b+c)s(s-a)+a(s-b)(s-c)-2bcs\\ =&(c+a)s(s-b)+b(s-c)(s-a)-2cas\\ =&(a+b)s(s-c)+c(s-a)(s-b)-2abs,\end{aligned}$$ where $2s=a+b+c$.

10. If $2s=a+b+c$, prove that

$$\begin{aligned}&a(s-b)(s-c)+b(s-c)(s-a)+c(s-a)(s-b)\\ =&a(s-a)^2+b(s-b)^2+c(s-c)^2\\ =&abc-2(s-a)(s-b)(s-c).\end{aligned}$$

11. Prove that

$$\begin{aligned}&(b+c-a-x)^4(b-c)(a-x)+(c+a-b-x)^4(c-a)(b-x)\\ &+(a+b-c-x)^4(a-b)(c-x)\\ &=16(b-c)(c-a)(a-b)(x-a)(x-b)(x-c).\end{aligned}$$

12. Show that $x^4+y^4+z^4-2x^2y^2-2y^2z^2-2z^2x^2$ is divisible by the four expressions $x\pm y\pm z$.

12. Let $F(x)$ denote a positive integral function of x, of the nth degree, say $p_n x^n + p_{n-1} x^{n-1} + \ldots . p_1 x + p_0$.

Let $a_1, a_2 \ldots . a_n$ be n different values of x, which make $F(x)=0$.

Then $F(x)$ is divisible by $x-a_1$; denote the quotient by Q_1, then $F(x)=Q_1(x-a_1)$.

Hence $Q_1(x-a_1)=0$, when $x=a_2$; but a_2-a_1 is not equal to 0, since a_2 and a_1 are different;

$\therefore$, [Art. 324], Q_1 must vanish when $x=a_2$;

$\therefore$ Q_1 is divisible by $x-a_2$; denote the quotient by Q_2;

$$\therefore \; F(x)=Q_2(x-a_1)(x-a_2).$$

Now Q_1 is a positive integral function of x of the $(n-1)$th degree, of which $p_n x^{n-1}$ is the first term, hence Q_2 is a similar function of the $(n-2)$th degree, of which the first term is $p_n x^{n-2}$. Proceeding thus, the co-efficient of the first term of each quotient being p_n, we at last come to

$$F(x)=p_n(x-a_1)(x-a_2)\ldots(x-a_n).$$

13. *Ex.* What are the limits of the values which x must have in order that $6x^2+7x-20$ may be positive?

Solving $6x^2+7x-20=0$ we find $x=-\frac{5}{2}$, or $\frac{4}{3}$;

$\therefore$, Art. 12, $6x^2+7x-20=6(x+\frac{5}{2})(x-\frac{4}{3})$.

(1) If x is greater than $\frac{4}{3}$, both the factors $x+\frac{5}{2}$, and $x-\frac{4}{3}$ are positive, and then their product is positive.

(2) If x is less than $-\frac{5}{2}$, both the factors are negative, and their product again is positive.

(3) If x is less than $\frac{4}{3}$, and greater than $-\frac{5}{2}$, then $x-\frac{4}{3}$ is negative, and $x+\frac{5}{2}$ is positive, and their product is negative.

Hence x must be greater than $\frac{4}{3}$, or less than $-\frac{5}{2}$, in order that $6x^2+7x-20$ may be positive, or in other words, $6x^2+7x-20$ is positive for all values of x, except those which lie between the roots of the corresponding equation.

14. *Ex.* Investigate the limits of the values that the expression $\frac{x^2+3x+4}{x^2+5x+2}$ can have, consistent with x being a real number.

$$\text{Put } \frac{x^2+3x+4}{x^2+5x+2}=u;$$

$$\text{then } x^2(1-u)+x(3-5u)+4-2u=0;$$

$$\therefore x=\frac{-(3-5u)\pm\sqrt{4(1-u)(2u-4)+(3-5u)^2}}{2(1-u)}$$

$$=\frac{5u-3\pm\sqrt{-7+17u^2-6u}}{2(1-u)};$$

$$\therefore 17u^2-6u-7 \text{ must be positive.}$$

Now if $17u^2-6u-7=0$,

$$u=\frac{3\pm\sqrt{119+9}}{17}=\frac{3\pm8\sqrt{2}}{17};$$

$\therefore$, Art. 12, $17u^2-6u-7=17\left(u-\frac{3+8\sqrt{2}}{17}\right)\left(u-\frac{3-8\sqrt{2}}{17}\right)$;

$\therefore$, as in Art. 13,

u must be greater than $\frac{3+8\sqrt{2}}{17}$, or less than $\frac{3-8\sqrt{2}}{17}$.

It will be remarked here that $17u^2-6u-7$ is positive so long as u does not lie between the roots of the corresponding equation. This statement and that at the end of Art. 13 are particular cases of the proposition investigated in the following Article.

15. PROP. *To show that for all real values of* x *the expression* ax^2+bx+c *has the same sign as* a, *except when the roots of the corresponding equation are real and different, and* x *lies between them.*

For $ax^2+bx+c=a\left\{x^2+\frac{bx}{a}+\frac{c}{a}\right\}$

$$=a\left\{x^2+\frac{bx}{a}+\frac{b^2}{4a^2}+\frac{c}{a}-\frac{b^2}{4a^2}\right\}=a\left\{\left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a^2}\right\},$$

and $\therefore$ has, or has not, the same sign as a, according as $\left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a^2}$ is positive, or negative.

Now $\left(x+\frac{b}{2a}\right)^2$, being a square, is always positive, whatever real value x may have.

1°. Let the roots of the equation $ax^2+bx+c=0$ be impossible, then b^2-4ac, and $\therefore \frac{b^2-4ac}{4a^2}$, is negative; $\therefore \left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a^2}$ is positive.

2°. Let the roots of the equation be real and equal, then $b^2-4ac=0$; $\therefore \left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a^2}$ becomes $\left(x+\frac{b}{2a}\right)^2$, and $\therefore$ is positive.

Hence in both these cases a and ax^2+bx+c have the same sign.

3°. Let the roots of the equation be real and different, then b^2-4ac, and $\therefore \frac{b^2-4ac}{4a^2}$, is positive.

Now $\left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a^2}$ is positive as long as $\left(x+\frac{b}{2a}\right)^2$ is greater than $\frac{b^2-4ac}{4a^2}$;

i.e., $x+\frac{b}{2a}$ is numerically greater than $\pm\frac{\sqrt{b^2-4ac}}{2a}$;

i.e., $x+\frac{b}{2a}$ does not lie between 0 and $\frac{\sqrt{b^2-4ac}}{2a}$, nor between 0 and $-\frac{\sqrt{b^2-4ac}}{2a}$;

i.e., $x+\frac{b}{2a}$ does not lie between $\frac{\sqrt{b^2-4ac}}{2a}$ and $-\frac{\sqrt{b^2-4ac}}{2a}$;

i.e., x does not lie between $\frac{-b+\sqrt{b^2-4ac}}{2a}$ and $\frac{-b-\sqrt{b^2-4ac}}{2a}$.

But these last expressions are the roots of the equation.

Hence in this case a and ax^2+bx+c have the same sign. except when x lies between the roots of the equation.

EXAMPLES.—III.

1. What values of x will render $3x^2-7x+2$ positive?

2. Find the least positive value which $\dfrac{7x^2+3x+1}{x^2+4x+1}$ can have.

3. When x is real, find the limits to the values of $\dfrac{2x^2-3x+1}{x+1}$, and $\dfrac{3x^2+7x-1}{x^2+3}$.

4. Find the greatest value which $\dfrac{x^2+10x+25}{x^2+5x+7}$ can have.

5. What values of x will make $4x-7x^2+1$ positive?

6. What values are possible for $\dfrac{x^2-4x+5}{x^2+4x+5}$, and $\dfrac{9x^2+9x+2}{3x^2+4x+2}$?

7. Determine the limits between which $\dfrac{x^2-3x-3}{2x^2+2x+1}$ lies for all possible values of x.

16. *Ex.* To find the sum of the squares of the first n natural numbers.

Let Σn^2 denote $1^2+2^2+\,.\,.\,n^2$.

Now
$$(n+1)^3=n^3+3n^2+3n+1,$$
$$n^3=(n-1)^3+3(n-1)^2+3(n-1)+1,$$
$$\text{etc.}=\text{etc.}$$
$$3^3=2^3+3.2^2+3.2+1,$$
$$2^3=1^3+3.1^2+3.1+1;$$

$\therefore$, adding,
$$(n+1)^3=1+3(n^2+..+2^2+1^2)+3(n+..+2+1)+n$$
$$=1+3\Sigma n^2+3\frac{n(n+1)}{2}+n;$$
$$\therefore\ \Sigma n^2=\frac{1}{3}\left\{(n+1)^3-(n+1)-3\frac{n(n+1)}{2}\right\}$$
$$=\frac{n+1}{6}\left\{2(n+1)^2-2-3n\right\}$$
$$=\frac{n+1}{6}\left\{2n^2+n\right\}$$
$$=\frac{n(n+1)(2n+1)}{6}.$$

17. *Ex.* To find the sum of the cubes of the first n natural numbers.

Let $\Sigma n^3 = 1^3 + 2^3 + \ldots n^3$.

Now $(n+1)^4 = n^4 + 4n^3 + 6n^2 + 4n + 1$,

$$n^4 = (n-1)^4 + 4(n-1)^3 + 6(n-1)^2 + 4(n-1) + 1,$$

etc. = etc.

$$2^4 = 1^4 + 4.1^3 + 6.1^2 + 4.1 + 1;$$

$\therefore$, adding, $(n+1)^4 = 1 + 4\Sigma n^3 + n(n+1)(2n+1) + 2n(n+1) + n$.

$$\therefore \Sigma n^3 = \tfrac{1}{4}\{(n+1)^4 - (n+1) - n(n+1)(2n+1) - 2n(n+1)\}$$
$$= \tfrac{1}{4}(n+1)\{(n+1)^3 - 1 - n(2n+1) - 2n\}$$
$$= \tfrac{1}{4}n(n+1)(n^2 + 3n + 3 - 2n - 1 - 2)$$
$$= \tfrac{1}{4}n^2(n+1)^2 = \left\{\frac{n(n+1)}{2}\right\}^2.$$

In the same way, by expanding $(n+1)^5$ etc., and adding, we might obtain an expression for Σn^4.

18. *Ex.* These expressions can be easily applied to obtain the sums of n terms of other series.

Thus $1.2 + 2.3 + 3.4 \ldots + n(n+1)$

$$= 1^2 + 1 + 2^2 + 2 + 3^2 + 3 + \ldots\ldots + n^2 + n$$
$$= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$$
$$= \frac{n(n+1)}{6}\{2n+1+3\}$$
$$= \frac{n(n+1)(n+2)}{3}.$$

EXAMPLES.—IV.

Sum to n terms the series—

1. $2.3 + 2.3.4 + 3.4.5 + \ldots\ldots\ldots + n(n+1)(n+2)$
2. $2^2 + 4^2 + 6^2 + \ldots\ldots\ldots\ldots\ldots\ldots + \overline{2n}|^2$
3. $1^2 + 3^2 + 5^2 + \ldots\ldots\ldots\ldots\ldots\ldots + (2n-1)^2$
4. $2^3 + 4^3 + 6^3 + \ldots\ldots\ldots\ldots\ldots\ldots + (2n)^3$
5. $1.3 + 2.4 + 3.5 + \ldots\ldots\ldots\ldots\ldots + n(n+2)$
6. $1.2 + 2^2.3 + 3^2.4 \ldots\ldots\ldots\ldots\ldots + n^2(n+1)$.

19. *Ex.* Since recurring decimals are Geometric Progressions, we can find an expression for their values.

$$\text{Thus } 2\cdot26\dot{9}3\dot{2}=2+\frac{269}{1000}+\frac{32}{100,000}+\frac{32}{10,000,000}+\text{etc.}$$
$$=2+\frac{269}{1000}+\frac{32}{100,000}\left\{1+\frac{1}{100}+\frac{1}{(100)^2}+\text{etc.}\right\}$$
$$=2+\frac{269}{1000}+\frac{32}{100,000}\cdot\frac{1}{1-\frac{1}{100}}$$
$$=2+\frac{269}{1000}+\frac{32}{100,000}\cdot\frac{100}{99}$$
$$=2+\frac{269}{1000}+\frac{32}{99,000}$$
$$=2+\frac{269\times99+32}{99,000}$$
$$=2+\frac{269(100-1)+32}{99,000}$$
$$=2+\frac{26932-269}{99,000}.$$

20. Generally, let P and Q denote respectively the sequences of digits in the non-recurring, and the recurring, parts of a decimal. Suppose there are p digits in P and q in Q.

$$\text{Then the decimal}=\frac{P}{10^p}+\frac{Q}{10^{p+q}}\left\{1+\frac{1}{10^q}+\frac{1}{10^{2q}}+\text{etc.}\right\}$$
$$=\frac{P}{10^p}+\frac{Q}{10^{p+q}}\cdot\frac{1}{1-\dfrac{1}{10^q}}$$
$$=\frac{P}{10^p}+\frac{Q}{10^p}\cdot\frac{1}{10^q-1}$$
$$=\frac{P.10^q+Q-P}{10^p(10^q-1)}.$$

Now $10^q=100\ldots$ to q cyphers, and $10^p=100\ldots$ to p cyphers, and $10^q-1=999\ldots$ to q nines; $\therefore$ $10^p(10^q-1)$ stands for q nines followed by p cyphers; and $P.10^q+Q$ represents the sequence of figures to the end of the first period. Hence the rule given in *Arithmetic*.

21. *To find the number of combinations of* n *things taken* r *at a time.*

Denote the n things by the letters $a, b, c, d, \ldots$ and the required number by $(n)_r$.

We can form all such combinations into n classes,—(1) those in which a stands first, (2) those in which b stands first, and so on; and the sum of the numbers in all these classes is $r(n)_r$. For every combination occurs r times, viz., once in each class in which any one of its component things stands first.

For instance, when $r=4$, the combination $abcd$ occurs in each of the first four classes.

Now every combination of the first class can be formed by placing a before one of the combinations of the $n-1$ things $b, c, d \ldots$, taken $r-1$ at a time; and every one of these latter combinations gives a different combination of the first class. Hence the number in this class $=(n-1)_{r-1}$. Similarly we can show that $(n-1)_{r-1}$ is the number in each of the n classes; $\therefore$ the sum of the numbers in all these classes $=n(n-1)_{r-1}$.

$$\text{Hence } r(n)_r = n(n-1)_{r-1};$$

$$\therefore (n)_r = \frac{n}{r}(n-1)_{r-1},$$

$$\text{and } (n-1)_{r-1} = \frac{n-1}{r-1}(n-2)_{r-2},$$

$$\text{etc.} \quad = \quad \text{etc.},$$

$$(n-r+2)_2 = \frac{n-r+2}{2}(n-r+1)_1,$$

$$\therefore, \text{ multiplying, } (n)_r = \frac{n(n-1)\ldots(n-r+2)}{\underline{|r}}(n-r+1)_1;$$

$$\text{but } (n-r+1)_1 = n-r+1;$$

$$\therefore (n)_r = \frac{n(n-1)\ \ldots\ (n-r+2)(n-r+1)}{\underline{|r}}.$$

22. *To find the number of permutations of* n *things taken* r *at a time.*

Denote the n things by the letters a, b, c, d, . . . and the required number by the symbol $(n)_r$.

We can form these $(n)_r$ permutations into n classes,—(1) those in which a stands first, (2) those in which b stands first, and so on; and $(n)_r$ is the sum of the numbers in all these classes.

Now every permutation of the first class can be formed by placing a before one of the permutations of the $\overline{n-1}$ things b, c, d, . . . taken $r-1$ at a time; and every one of these latter permutations gives a different permutation of the first class. Hence the number in this class $=(n-1)_{r-1}$. Similarly it can be shown that $(n-1)_{r-1}$ is the number in each of the n classes; $\therefore$ the sum of the numbers in all these classes $=n(n-1)_{r-1}$.

$$\begin{aligned}
\text{Hence } (n)_r &= n(n-1)_{r-1},\\
\text{and } (n-1)_{r-1} &= (n-1)(n-2)_{r-1},\\
\text{etc.} &= \text{etc.};\\
(n-r+2)_2 &= (n-r+2)(n-r+1)_1,
\end{aligned}$$

$$\therefore, \text{ multiplying, } (n)_r = n(n-1)\ldots(n-r+2)(n-r+1)_1;$$

$$\text{but } (n-r+1)_1 = n-r+1;$$

$$\therefore (n)_r = n(n-1)\ldots(n-r+1).$$

I

On the Fundamental Operations of Algebra.

1. Arithmetical Algebra is the science which treats of the operations of arithmetic and their results, when they are performed on, and with, arithmetical numbers, these being represented by symbols which stand, not each for some one number, but for any which will *allow of the operations indicated being performed.*

Thus 5 can only stand for the number five, and $\frac{3}{4}$ for the number three-fourths, so that $5+\frac{3}{4}$ can only express the sum of five and three-fourths; but a and b can stand for any two arithmetical numbers, so that $a+b$ can express the sum of any two such numbers.

In Arithmetical Algebra we have then the consideration of such general theorems as are capable of proof in arithmetic for every particular case, which can be obtained by giving some one value to each of the symbols employed in the corresponding general theorem.

2. In some expressions we must understand that a certain *limitation* is imposed on a symbol, as to the numbers for which it can stand, on account of the proviso above mentioned *that they must be such as to allow of the operations indicated being performed.*

Thus in the expression $a-b$, a may represent *any* arithmetical number we please; but whatever that number may be, b can only represent some number not greater than a, for otherwise the operation we are directed to perform,—viz., the subtraction of

b from a, would be impossible in arithmetic. Again, in x^m, m can only represent whole numbers.

Again, $a+b-c=a-c+b$, is a theorem which can be proved for each set of particular values we can give to a, b, c, provided the value we give to c is not greater than the one we give to a. Hence it is true for all values of a, b, c, except when c is greater than a.

Also, $x^m \div x^n = x^{m-n}$ can only be proved in arithmetic when m and n stand for *whole* numbers, and n is less than m.

3. With such limitations as the above, we have the following theorems (or laws) in Arithmetical Algebra:—

I. The Commutative Law.

(1) Additions and subtractions may be performed in any order. *Ex.* $a+b+c-d=a+b-d+c=a-d+b+c=b-d+a+c=$ etc. This is generally considered self-evident, since the operations of addition and subtraction are not confined to arithmetic.

(2) Multiplications and divisions may be performed in any order.

Ex. $a\times b\times c=b\times a\times c=a\times c\times b=c\times a\times b$. [Art. 39].

Also $a\times b\div c=a\div c\times b=b\div c\times a=$ etc.

For $a\times b\div c=a\times b\times\dfrac{1}{c}=\dfrac{ab}{c}$,

and $a\div c\times b=a\times\dfrac{1}{c}\times b=\dfrac{ab}{c}$.

Similarly we can show that $b\div c\times a=\dfrac{ab}{c}$, and $\therefore$ the operations indicated by $a\times b\div c$, etc., give the same result.

II. The Distributive Law.

(1) Additions and subtractions of numbers may be distributed over a series of additions and subtractions of their parts.

Ex. $a+b+(c+d-e)=a+b+c+d-e$.

$a+b-(c+d-e)=a+b-c-d+e$. [Art. 18].

(2) Multiplications (and divisions) of numbers by one another may be distributed over a series of additions and subtractions of the products (and quotients) of their parts.

Ex. $(a-b)(c-d)=ac-bc-ad+bd.$

$(a+b)(c-d)=ac+bc-ad-bd.$ [Art. 48-55.]

Also $(a+b-c)\div x=a\div x+b\div x-c\div x.$

For let $q_1=a\div x, q_2=b\div x, q_3=c\div x$;

$\therefore a=q_1x, b=q_2x, c=q_3x$; [Art. 72.]

$\therefore a+b-c=q_1x+q_2x-q_3x=(q_1+q_2-q_3)x$, by *Ex.* above;

$\therefore (a+b-c)\div x=q_1+q_2-q_3.$

III. **Laws of Exponents.**

(1) $x^m\times x^n=x^{m+n}.$ [Art. 269.]

(2) $x^m\div x^n=x^{m-n}$, if $m>n$

$=1,$ if $m=n$

$=\dfrac{1}{x^{n-m}}$, if $m<n$.

This can as easily be proved as (1) is in [Art. 269].

(3) $(x^m)^n=x^{mn}$ [Art. 269].

(4) $(xy)^m=x^my^m$ [Art. 288].

4. In Symbolical Algebra we consider the symbols, both of number* and of operation, as capable of having any meaning that will permit of their obedience to the fundamental laws of Arithmetical Algebra.

5. Suppose then, first of all, that a and b represent numbers, but remove the limitation that in the expression $a-b$, b is not greater than a. Put $a=0$, $b=6$ and we have $0-6$.

This is a result we have never met with before; we must therefore invent a new symbol for it. Denote it by -6. Generally we define $-b$ to mean $0-b$, and call it a negative number or symbol.

* By a symbol of number, we mean such a symbol as up to this point we have used to represent numbers.

6. By analogy we should denote $0+b$ by $+b$, and call it a positive number or symbol.

$$\text{Now } +b=0+b=b+0, \text{ Art. 3, I. 1.,}$$
$$=b.$$

Thus positive numbers, and arithmetical numbers, are the same. Indeed this was anticipated in [Art. 12], where it was stated that "when no symbol precedes a term the symbol $+$ is understood."

7. In the symbol $-b$, $-$ is called the sign of *affection*, and b is called the magnitude of the number. Thus $-b$ and $+b$ are said to be equal in magnitude and opposite in sign, so that to every negative there is a corresponding positive number, equal in magnitude and opposite in sign, and also a corresponding arithmetical number.

8. We must now investigate what meanings must be attached to the symbols of operation, when used with negative numbers.

Addition and Subtraction.

$$\begin{aligned} a+(-b)&=a+(0-b), \text{ by definition,} \\ &=a+0-b, \qquad \text{Law I. 1.} \\ &=a-b. \\ a-(-b)&=a-(0-b), \\ &=a-0+b, \\ &=a+b. \end{aligned}$$

Hence to add, or subtract, a negative, is the same as to subtract, or add, the corresponding positive number.

9. Again, $-b\pm a=0-b\pm a=0-(b\mp a)=-(b\mp a)$.

Hence to add, or subtract, a positive, to, or from, a negative number is to decrease, or increase, the magnitude of the negative by that of the positive number.

10. *Multiplication* with negative numbers has been explained in [Art. 58].

Thence it follows that $-a=(-1)a$, and $(-a)^m=\pm a^m$, according as m is an even, or odd, integer. Also if $a=-b$, $-a=(-1)(-b=b)$.

11. *Division.*

Let q denote the quotient $-a \div b$;

$$\therefore\ -a = bq; \qquad \text{[Def. Art. 72]}$$
$$\therefore\ a = -bq \qquad \text{Art. 10}$$
$$= b(-q); \qquad \text{[Art. 58]}$$
$$\therefore\ -q = a \div b;\ \therefore\ q = -(a \div b).$$

Hence to divide a negative by a positive number is the same as to divide the positive number corresponding to the dividend by the divisor, and prefix the sign $-$ to the quotient.

Again let q denote $a \div (-b)$;

$$\therefore\ a = q(-b) = -bq = b(-q);$$
$$\therefore\ -q = a \div b;\ \therefore\ q = -(a \div b).$$

Hence to divide by a negative number is the same as to divide by the corresponding positive number, and prefix the sign $-$ to the quotient.

12. We give no *à priori* definition of an operation performed with, or on, a negative symbol, but deduce its meaning from the principle that *the result of such an operation is to be the same in form as if the symbols operated on had represented arithmetical numbers.*

Thus we gave no definition of the operation of addition of a negative number, but from the principle that

$$a + (c - b) = a + c - b, \text{ whether } b \text{ is} > c \text{ or not,}$$

we found that it must mean, such an operation that its result is the same as subtracting the corresponding positive number. And any meaning that may hereafter be assigned to a negative number must allow of this meaning being true; accordingly we find in [Art. 35] that if we represent a gain by 200, the loss equal in amount is represented by -200.

13. The fraction $\dfrac{x}{y}$ has no meaning if either, or both, of x and y are negative. We must, in this case, extend the meaning of the symbol. It has been shown in [Art. 158] that, when x and y are positive, $\dfrac{x}{y} = x \div y$. Let us then define $\dfrac{x}{y}$ to represent the quotient $x \div y$, whatever x and y may be.

Therefore

$\frac{-a}{b} = -a \div b = -(a \div b)$, Art. 11, $= -\frac{a}{b}$ $\left\{\begin{array}{l}\text{where } -a \text{ is negative} \\ \text{and } b \text{ is positive.}\end{array}\right.$

$\frac{a}{-b} = a \div (-b) = -(a \div b) = -\frac{a}{b}$ $\left\{\begin{array}{l}\text{where } a \text{ is positive and} \\ -b \text{ negative.}\end{array}\right.$

$$\frac{-a}{-b} = -a \div (-b) = -(-a \div b) = -\{-(a \div b)\} = a \div b = \frac{a}{b},$$

where $-a$ and $-b$ are both negative.

From these all the theorems relating to fractions with positive terms can be proved for those in which either, or both, of the terms are negative.

14. The meaning and laws of combination of x^m, when the limitation as to m is so far removed as to allow of its representing a positive, or a negative, number have been already discussed in [Chap. XXIII.]

15. Positive and negative numbers are classed together as *real* numbers. That is to say, any symbol composed of an arithmetical number, preceded by the sign $+$, or by the sign $-$, is called a real number.

16. Since $(+b)(+b) = b^2$, and $(-b)(-b) = b^2$; therefore b^2 has two square roots,—viz., (1) $+b$, or b, which is the one we have been accustomed to in Arithmetic; and (2) $-b$, which is equal in magnitude to the arithmetical root, but opposite in sign.

17. Since the squares of all real numbers are positive, a real number cannot be the square root of a negative number. Hence $\sqrt{-b^2}$, or $(-b^2)^{\frac{1}{2}}$, where $-b^2$ is a negative number, is something we have not previously met with. Call it an *imaginary* number; it is such that $(\sqrt{-b^2})^2 = -b^2$.

This does not accurately define the meaning of $\sqrt{-b^2}$, or of the operation $(\ )^2$ performed on $\sqrt{-b^2}$. All we say is that $\sqrt{-b^2}$, and the operation, must be such, that the performance of one on the other must give $-b^2$ as the result.

18. We can put $\sqrt{-b^2}$ into another form.

For $\sqrt{-b^2}=\sqrt{(-1)b^2}=(-1)^{\frac{1}{2}}(b^2)^{\frac{1}{2}}$. Law III. 4.

$=(-1)^{\frac{1}{2}}(\pm b)=\pm b(-1)^{\frac{1}{2}}$ *i.e.* $\pm b\sqrt{-1}$.

19. Any expression involving the symbol $\sqrt{-1}$ is called an *impossible*, or *imaginary*, expression; and any expression which does not involve this symbol is said to be *real.*

20. In the symbols $+b\sqrt{-1}$ and $-b\sqrt{-1}$, b is generally supposed to be real, and is then called the magnitude of the symbol; also $+b\sqrt{-1}$ is called a positive, whilst $-b\sqrt{-1}$ is called a negative, imaginary number.

We shall always understand $b\sqrt{-1}$ to mean the same as $+b\sqrt{-1}$.

21. Prop. *If* a+b $\sqrt{-1}=0$, *and* a *and* b *are real, then* a=0 *and* b=0.

For by Law I. 1, $a+b\sqrt{-1}=b\sqrt{-1}+a$;

$$\begin{aligned}
&\therefore b\sqrt{-1}+a=0;\\
&\therefore b\sqrt{-1}+a-a=-a;\\
&\therefore b\sqrt{-1} \qquad =-a;\\
&\therefore (b\sqrt{-1})^2 \qquad =(-a)^2;\\
&\therefore -b^2 \qquad =a^2;\ \therefore a^2+b^2=0.
\end{aligned}$$

Now if either of the two positive numbers, a^2 and b^2, be of appreciable magnitude, their sum cannot be zero. Hence each is zero;

$$\therefore a^2=0,\ b^2=0; \qquad \therefore a=0,\ b=0.$$

Cor. *If* a, b, α, β *are real, and* a+b $\sqrt{-1}=\alpha+\beta\sqrt{-1}$ *then* a$=\alpha$, b$=\beta$.

For since $a+b\sqrt{-1}=\alpha+\beta\sqrt{-1}$,

$$\begin{aligned}
0&=a+b\sqrt{-1}-(\alpha+\beta\sqrt{-1})\\
&=a+b\sqrt{-1}-\alpha-\beta\sqrt{-1}, && \text{Law II. 1.}\\
&=a-\alpha+b\sqrt{-1}-\beta\sqrt{-1}, && \text{Law I. 1.}\\
&=a-\alpha+(b-\beta)\sqrt{-1}; && \text{Law II. 2.}
\end{aligned}$$

$\therefore a-\alpha=0$, and $b-\beta=0$, by the Prop.,
or $a=\alpha$, and $b=\beta$.

22. The following are examples of the ways in which we can combine imaginary numbers with one another and with real numbers.

By Art. 13, $\dfrac{a}{b\sqrt{-1}}=a\div b\sqrt{-1},=q$ say;

$$\therefore\ a=qb\sqrt{-1};$$

$$\therefore\ a\sqrt{-1}=qb(\sqrt{-1})^2=-qb;$$

$$\therefore\ (-1)a\sqrt{-1}=qb;$$

$$\therefore\ q=\frac{(-1)a\sqrt{-1}}{b}=,\ \text{Art. 10},\ \frac{-a\sqrt{-1}}{b}=-\frac{a}{b}\sqrt{-1}.$$

Similarly $\dfrac{a\sqrt{-1}}{b\sqrt{-1}}=\dfrac{a}{b}$.

Again $(a\sqrt{-1})^x=a^x(\sqrt{-1})^x=a^x(-1)^{\frac{x}{2}}$;

and $(a+b\sqrt{-1})\,(c-d\sqrt{-1})$

$=ac+b\sqrt{-1}\,c-ad\sqrt{-1}-b\sqrt{-1}\,d\sqrt{-1}$, Law II. 2.

$=ac+bc\sqrt{-1}-ad\sqrt{-1}-bd\sqrt{-1}\sqrt{-1}$, " I. 2.

$=ac+bc\sqrt{-1}-ad\sqrt{-1}-bd\,(-1)$, Def. Art. 17.

$=ac+bc\sqrt{-1}-ad\sqrt{-1}+bd$, [Art. 58.]

$=ac+bd+(bc-ad)\sqrt{-1}$.

23. Also $\dfrac{ma}{m(c+d\sqrt{-1})}=\dfrac{a}{c+d\sqrt{-1}}$.

For let $\dfrac{a}{c+d\sqrt{-1}}=q$; $\therefore\ a=q(c+d\sqrt{-1})$;

$$\therefore\ ma=mq(c+d\sqrt{-1})$$
$$=qm(c+d\sqrt{-1});$$

$$\therefore\ q=\frac{ma}{m(c+d\sqrt{-1})}.$$

So $\dfrac{a}{c+d\sqrt{-1}}=\dfrac{a(c-d\sqrt{-1})}{(c+d\sqrt{-1})\,(c-d\sqrt{-1})}=\dfrac{ac-ad\sqrt{-1}}{c^2-(d\sqrt{-1})^2}$

$$=\frac{ac-ad\sqrt{-1}}{c^2+d^2}.$$

Also $\dfrac{a}{c+d\sqrt{-1}}+\dfrac{a}{c-d\sqrt{-1}}=\dfrac{ac-ad\sqrt{-1}}{c^2+d^2}+\dfrac{ac+ad\sqrt{-1}}{c^2+d^2}$

$$=\frac{ac+ac+ad\sqrt{-1}-ad\sqrt{-1}}{c^2+d^2}$$

$$=\frac{2ac}{c^2+d^2}.$$

Again

$$(a+b\sqrt{-1})^m=a^m+ma^{m-1}b\sqrt{-1}+\frac{m(m-1)}{1.2}a^{m-2}(b\sqrt{-1})^2+$$

$$\frac{m(m-1)(m-2)}{1.2.3}a^{m-3}(b\sqrt{-1})^3+\text{etc.}$$

$$=a^m+ma^{m-1}b\sqrt{-1}-\frac{m(m-1)}{1.2}a^{m-2}b^2-$$

$$\frac{m(m-1)(m-2)}{1.2.3}a^{m-3}b^3\sqrt{-1}+\text{etc.}$$

$$=a^m-\frac{m(m-1)}{1.2}a^{m-2}b^2+\text{etc.}$$

$$+\left(ma^{m-1}b-\frac{m(m-1)(m-2)}{1.2.3}a^{m-3}b^3+\text{etc.}\right)\sqrt{-1}.$$

Thus $(a+b\sqrt{-1})^m$ has been reduced to the form $A+B\sqrt{-1}$, where A and B are real.

24. *To put* $\sqrt{\mathrm{a}+\mathrm{b}\sqrt{-1}}$ *into the form* $\alpha+\beta\sqrt{-1}$.

Let $\sqrt{a+b\sqrt{-1}}=\sqrt{x}+\sqrt{y}\sqrt{-1}$;

$\therefore\ a+b\sqrt{-1}=x-y+2\sqrt{xy}\sqrt{-1}$;

$\therefore\ a=x-y,\ b=2\sqrt{xy}$, Art. 21. Cor.;

$$\therefore\ x=\frac{a}{2}\pm\frac{\sqrt{a^2+b^2}}{2},$$

and $$y=-\frac{a}{2}\pm\frac{\sqrt{a^2+b^2}}{2};$$

$$\therefore\ \sqrt{a+b\sqrt{-1}}=\left(\frac{a}{2}\pm\frac{\sqrt{a^2+b^2}}{2}\right)^{\frac{1}{2}}+\left(-\frac{a}{2}\pm\frac{\sqrt{a^2+b^2}}{2}\right)^{\frac{1}{2}}\sqrt{-1},$$

both the upper, or both the lower, signs being taken.

25. A system has been devised by which imaginary expressions represent the *positions*, as well as the lengths of lines, so as still to obey all the fundamental Laws of Algebra. For explanation of this system the reader is referred to the works of Warren, Peacock, and De Morgan.

EXAMPLES.—I.

1. Express $(a+b\sqrt{-1})^2$, and $(a-b\sqrt{-1})^3$ in the form $A+B\sqrt{-1}$.

2. Rationalize the denominators in

$$\frac{1}{a-b\sqrt{-1}},\ \frac{x-y\sqrt{-1}}{x+y\sqrt{-1}},\ \frac{x+y\sqrt{-1}}{\sqrt{-1}.}.$$

3. Simplify $\dfrac{a+b\sqrt{-1}}{a-b\sqrt{-1}}+\dfrac{a-b\sqrt{-1}}{a+b\sqrt{-1}}$.

4. Find the value of $(2+3\sqrt{-5})^{\frac{1}{2}}+(2-3\sqrt{-5})^{\frac{1}{2}}$.

5. Find the values of $\left(\dfrac{-1+\sqrt{-3}}{2}\right)^2$, and $\left(\dfrac{-1+\sqrt{-3}}{2}\right)^3$.

6. Find the square root of $-79-8\sqrt{-5}$.

7. Express in a form free from imaginaries the square of

$$\sqrt[4]{a+b\sqrt{-1}}+\sqrt[4]{a-b\sqrt{-1}}.$$

8. Reduce $\dfrac{1+\sqrt{-1}}{1-\sqrt{-1}}$; $\dfrac{5-\sqrt{-2}}{1+\sqrt{-2}}$.

9. Prove that $\sqrt{3+4\sqrt{-1}}+\sqrt{3-4\sqrt{-1}}=4$.

10. Reduce to a real surd the expression

$$\sqrt{3+2\sqrt{-1}}+\sqrt{3-2\sqrt{-1}}.$$

II

On the Symbols ∞ and 0, and the word "Limit."

26. The symbol ∞ indicates that the symbol with which it is connected represents a magnitude, or ratio, which is being *endlessly increased;* thus "x is ∞" means that the magnitude, or ratio, represented by x is endlessly increased, or, as it is sometimes expressed, is increased without limit.

Obs. 1. Thus the learner will understand that by saying that x is ∞ we do not mean that x is to have any precise value, as we do when we say that x is 2; we only mean that there is no end to the increase of the magnitude, or ratio, represented by x.

Obs. 2. The word "*infinite*" is often used instead of the words "endlessly increased," and had better be understood to mean exactly the same thing. By an infinite number we mean a number which is endlessly increased.

Obs. 3. The symbol $=$ is often placed before ∞, instead of "is," or some other similar verb. Thus we have "$x=\infty$," which is read "x is infinite," or less correctly "x equals infinity." And we often find, "when $x=\infty$," used instead of the more correct phrase, "as x is endlessly increased," as if there were some precise point at which x ceased to be finite and became *in*finite.

Obs. 4. This abbreviation is likely to lead to great confusion, unless its precise meaning is carefully remembered.

Obs. 5. Thus when $x=2$ and $y=2$, we at once infer, as in Euc. Ax. I., that $x=y$. Similarly when $x=\infty$ and $y=\infty$, the learner

is apt to think that $x=y$, forgetting that $=$ before ∞ has not the same meaning as $=$ before 2, or any other symbol of finite number. Take another instance, although as $x=\infty$, $4x=\infty$ also, yet x and $4x$ do not tend to equality, or in other words, $x:4x$ does not tend to a ratio of equality, for it always remains the same as $1:4$.

27. The symbol 0 means, as stated at the beginning of Arithmetic, absolute *nonentity*, or the total absence of any magnitude, such as in any statement may be denoted by the symbols of number.

The symbol 0 has also another use. It indicates that the symbol, with which it is connected, represents a magnitude, or ratio, which is being endlessly decreased; thus "x is 0," means that x represents a magnitude, or ratio, which is endlessly decreased, or, as it is sometimes expressed, is decreased without limit.

Observations similar to 1, 2, 3, 4 of Art. 26 apply to 0 when used in this sense, only reading "decrease" for "increase," "zero" for "infinite" and "infinity," "0" for "∞."

Obs. 5. Further, when $x-y=0$, if we know that x and y represent finite numbers, we can infer that $x=y$; but if both $x=0$ and $y=0$, then we cannot infer that $x=y$, for since x and y are now endlessly decreased, their difference is also, whether they tend to equality or not.

Hence the only general test of x being equal to y, whether x and y are endlessly decreased, or increased, or are finite, is $\frac{x}{y}=1$, or $\frac{y}{x}=1$.

28. We will exemplify our statements by remarks on some of the connexions between 0 and ∞.

When $x=0$, $\frac{a}{x}=\infty$. This means that, as x is endlessly de-

creased, the fraction $\frac{a}{x}$ is endlessly increased. We also express this by saying that $\frac{a}{x}$ has *no limit*, or is infinite, when x is zero.

Conversely when $x=\infty$, $\frac{a}{x}=0$. This means that as x is endlessly increased, $\frac{a}{x}$ is endlessly decreased, in other words, that as x increases without limit, $\frac{a}{x}$ decreases without limit.

29. Again if $a<1$, $a^x=0$ when $x=\infty$. This means that as x is endlessly increased, a^x is endlessly decreased, or decreases without limit.

Consider the expression $b+a^x$, where $a<1$; when $x=\infty$, *i.e.* is endlessly increased, $a^x=0$, *i.e.* is endlessly decreased, and $b+a^x$ approaches endlessly near to b.

We sometimes express the same thing thus, if $a<1$, $b+a^x$ has b for its limit, when $x=\infty$, by which we mean, that as x increases the value of $b+a^x$ approaches b, and by taking x sufficiently great, the difference between it and b can be endlessly decreased, or in other words, can be made infinitely small.

In the same way a^x, *i.e.* $0+a^x$, is often said to have 0 for its limit, when $x=\infty$, *i.e.* the difference between a^x and 0 can be made infinitely small by endlessly increasing x. But if the 0 here spoken of have the second meaning, viz., of endless decrease, a^x has no precise value, and cannot be spoken of properly as a limit. If, on the other hand, 0 is here supposed to have the meaning of *nonentity*, it is to be observed that a^x, when $x=\infty$, represents *some*thing, though endlessly small, and there is therefore, as it were, a difference in kind between it and the symbol of absolute nonentity.

It would therefore seem advisable to content ourselves with the statement that $a^x=0$ when $x=\infty$, giving to these phrases the meanings assigned in Art. 26, 27.

30. If in the product $\alpha\beta$, one of the factors, as α, is zero, whilst β is zero or finite, then the product vanishes; but if α is zero whilst β is infinite, it does not necessarily follow that $\alpha\beta$ is zero.

For take the following simple examples:—Let $\alpha=v^2$, and put

(1.) $\beta=\frac{1}{v}$, then $\alpha\beta=v$; $\therefore$ when $\alpha=0$, *i.e.* $v=0$, $\beta=\infty$ and $\alpha\beta=0$;

(2.) $\beta=\frac{1}{v^2}$, then $\alpha\beta=1$; $\therefore$ when $\alpha=0$, *i.e.* $v=0$, $\beta=\infty$ and $\alpha\beta=1$;

(3.) $\beta=\frac{1}{v^3}$, then $\alpha\beta=\frac{1}{v}$; $\therefore$ when $\alpha=0$, *i.e.* $v=0$, $\beta=\infty$ and $\alpha\beta=\infty$.

31. We will now explain more fully the meaning of the word limit which we have used above.

Consider, for example, the expression $\frac{2+3x}{3+2x}$, call it A.

When $x=1$, A has a precise value, namely 1, and so it has for every other finite value of x except $-\frac{3}{2}$, which we shall speak of later. Next let $x=0$, if by this we mean that x represents absolute nonentity, then A has a precise value, namely $\frac{2}{3}$; but if by $x=0$, we mean that x is to represent a continually decreasing magnitude or ratio, then A cannot be said to have any precise value, for x has not, but the more x is decreased the more nearly does A approach to $\frac{2}{3}$, and by endlessly decreasing x we can endlessly decrease the difference between A and $\frac{2}{3}$, so that $\frac{2}{3}$ is the *limit of the values* which A has, as x is endlessly decreased.

Again, when $x=\infty$, A cannot be said to have any precise value, but $A=\dfrac{3+\frac{2}{x}}{2+\frac{3}{x}}$; $\therefore$ as x is increased, the more nearly does

A approach to $\frac{3}{2}$, and by endlessly increasing x we can endlessly decrease the difference between A and $\frac{3}{2}$; hence $\frac{3}{2}$ is the *limit of the values* which A has, as x is endlessly increased.

When $x=-\frac{3}{2}$, $A=\infty$, *i.e.* the value of A is endlessly increased as x approaches $-\frac{3}{2}$, and therefore A cannot be said to have then any precise value, nor is there any limit to the various values which A has then.

III

On Inequalities.

32. In this Chapter we shall discuss a number of propositions, which have for their object to prove that one of two given expressions is greater, or less, than another.

33. *Definition.* An expression (a) is said to be greater than another (b), when the difference $a-b$ is positive, and less if $a-b$ is negative.

34. We use the symbol $>$ for the words "is greater than," and $<$ for "is less than."

Thus $5>3$, $\because 5-3=+2$; but $-5<-3$, $\because -5-(-3)=-2$; and, generally, if $a>b$, then $a-b$ being positive, $b-a$, or $-a-(-b)$, is negative, and $\therefore -a>-b$.

Again $0>-7$, for $0-(-7)=+7$.

35. The statements $a>b$, and $a<b$, are called *inequalities.*

The statement $a>b>c$ means that $a>b$, that $b>$, or $=$, c, and that therefore $a>c$.

The statement $a \gtrless b$ is to be read "a is greater or less than b."

36. If $a>b$, and if m is positive, then $ma>mb$, for $ma-mb$, *i.e.*, $m(a-b)$, is positive, for both m and $a-b$ are positive; but if m is negative, then $m(a-b)$ is negative, and $\therefore ma<mb$.

Similarly, if $a<b$, $ma \gtrless mb$, according as m is negative or positive.

Again $a+m \gtrless b+m$ according as $a \gtrless b$ whatever m may be, positive or negative.

37. If a and b are real, $a^2-2ab+b^2=(a-b)^2$; but we know that the square of every real quantity is positive; $\therefore\ a^2+b^2-2ab$ is positive, hence $a^2+b^2>2ab$.

Similarly if x and y are both positive, $x+y>2\sqrt{xy}$, since $(\sqrt{x}-\sqrt{y})^2$ is positive.

38. We can apply these considerations to prove a great number of inequalities.

Ex. 1. $a^2+b^2+c^2>bc+ca+ab$.

$$\text{For } \left.\begin{array}{r} a^2+b^2>2ab \\ b^2+c^2>2bc \\ c^2+a^2>2ca \end{array}\right\} \therefore\text{, adding, } 2(a^2+b^2+c^2)>2(ab+bc+ca); \quad \therefore\ a^2+b^2+c^2>bc+ca+ab.$$

Note.—If $a=b=c$ this inequality becomes an equality.

Similarly most inequalities become equalities for special values of the symbols involved.

Ex. 2. The sum of any positive number (a) and its reciprocal >2.

$$\text{For } a+\frac{1}{a}=\frac{a^2+1}{a}>\frac{2a}{a}>2.$$

Ex. 3. If a, b, c are real numbers, not all equal, then $(b-c)(c-a)+(c-a)(a-b)+(a-b)(b-c)$ is negative.

For the expression $=2\{bc+ca+ab-(a^2+b^2+c^2)\}$, and this is negative, since by *Ex.* 1. $bc+ca+ab<a^2+b^2+c^2$.

Ex. 4. If x and y are positive, $x^5+y^5-x^4y-xy^4$ is positive. For the expression $=x(x^4-y^4)-y(x^4-y^4)=(x-y)(x^4-y^4)$ $=(x^2+y^2)(x+y)(x-y)^2$, and this is positive, since each of the factors composing it is positive.

Ex. 5. If a, b, c are positive, and not all equal, prove that $a(a-b)(a-c)+b(b-c)(b-a)+c(c-a)(c-b)$ is positive.

Let a, b, c be in descending order of magnitude.

The expression $=(a-b)(a^2-ac-b^2+bc)+c(c-a)(c-b)$ $=(a-b)^2(a+b-c)+c(c-a)(c-b)$, and this is positive since $a+b>c$, and $c-a$, $c-b$ are both negative.

39. The attention of the student is drawn to the following statements; he will immediately perceive their truth.

(1) If $a>b$ and $c>d$, then $a+c>b+d$, and $ac>bd$.

(2) If $a>b$ and $c<d$, then $a-c>b-d$, and $\frac{a}{c}>\frac{b}{d}$.

(3) If $a>b$, a and b being positive, when n is a positive integer, then $a^n>b^n$, and $a^{-n}<b^{-n}$; but when n is fractional, the character of the inequality will depend on the signs we affix to the roots.

Thus if $a=3$, $b=2$, $n=\frac{1}{2}$, then $\sqrt{3}>\sqrt{2}$, whilst $-\sqrt{3}<\sqrt{2}$, $-\sqrt{3}<-\sqrt{2}$, and $\sqrt{3}>-\sqrt{2}$.

40. *Ex.* 1. If a and b are positive, and $a>b$, find the limits to the value of x when $\dfrac{x+a}{\sqrt{a^2+x^2}}>\dfrac{x+b}{\sqrt{b^2+x^2}}$.

$$\text{By Art. 39 (3)}\quad \frac{(x+a)^2}{a^2+x^2}>\frac{(x+b)^2}{b^2+x^2};$$

$$\therefore 1+\frac{2ax}{a^2+x^2}>1+\frac{2bx}{b^2+x^2};$$

$$\therefore \frac{ax}{a^2+x^2}>\frac{bx}{b^2+x^2};$$

$$\therefore ab^2x+ax^3>a^2bx+bx^3;$$

$$\therefore x(a-b)(x^2-ab) \text{ is positive};$$

but $(a-b)$ is positive, $\therefore$ $x(x-\sqrt{ab})(x+\sqrt{ab})$ is positive; $\therefore$ if x is negative it must not be numerically greater than $\sqrt{ab}$, and if positive it must not be less than $\sqrt{ab}$;
$\therefore$ x must be between 0 and $-\sqrt{ab}$, or be greater than $\sqrt{ab}$.

See Appendix to Part I. Art. 13 and 14.

Ex. 2. Show that $(n+1)^{n-1} < \{\underline{|n}\}^2$.

We have $n+1 < r(n-r+2)$,

if $r^2-(n+2)r+n+1$ is negative,

i.e., if $(r-\overline{n+1})(r-1)$ is negative,

i.e., if r lies between 1 and $n+1$.

Hence, putting r successively equal to 2, 3, . . . n, we see that

$$n+1<2.n,$$
$$n+1<3.(n-1),$$
$$\text{etc.} < \text{etc.},$$
$$n+1<n.2,$$

and there being $n-1$ of these inequalities, we have by multiplying

$$(n+1)^{n-1}<\{\underline{|n}\}^2.$$

41. *If* $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots \frac{a_n}{b_n}$ *be fractions, which are not all equal, then* $\frac{a_1+a_2+\ldots+a_n}{b_1+b_2+\ldots+b_n}$ *lies between the least and greatest of them.*

Change the signs of the numerator and denominator of any one of which the denominator is negative, so that we may take all the denominators as positive.

Let $\frac{a_r}{b_r}$ be the least of the fractions, denote it by λ;

$$\therefore \frac{a_1}{b_1}>\lambda, \frac{a_2}{b_2}>\lambda, \ldots \frac{a_r}{b_r}=\lambda, \ldots \frac{a_n}{b_n}>\lambda;$$

then since $b_1, \ldots b_n$ are all positive,

$$a_1>\lambda b_1,\ a_2>\lambda b_2,\ \ldots\ a_r=\lambda b_r,\ \ldots\ a_n>\lambda b_n, \text{ Art. 36};$$

$$\therefore a_1+a_2+\ldots+a_n>\lambda(b_1+b_2+\ldots b_n), \text{ Art. 39, I};$$

$$\therefore \frac{a_1+a_2+\ldots+a_n}{b_1+b_2+\ldots+b_n}>\lambda>\frac{a_r}{b_r}.$$

Similarly it can be shown that $\frac{a_1+a_2+\ldots+a_n}{b_1+b_2+\ldots+b_n}$ is less than the greatest of the fractions.

EXAMPLES.—II.

Prove that the following inequalities are generally true, stating the cases of exception, the letters denoting positive numbers except when otherwise indicated.

1. $a^4b^2+a^2b^4>2a^3b^3$; $(a+b)^2>4ab$.

2. $(-a+b+c)^2+(a-b+c)^2+(a+b-c)^2>bc+ca+ab$.

3. $a^3+b^3+c^3>3abc$.

4. $\dfrac{b+c}{a}+\dfrac{c+a}{b}+\dfrac{a+b}{c}>6$.

5. $x^3+1>x^2+x$, if $x+1$ be positive; and $x^3-1>x^2-x$, if $x>1$.

6. $a^3+3ab^2 \gtreqless b^3+3ba^2$ according as $a \gtreqless b$.

7. $ac+bd>2\sqrt{abcd}$, $ab+dc>2\sqrt{abcd}$, etc.

8. $a^2cd+b^2ad+c^2ab+d^2bc>4abcd$.

9. $(a_1+a_2+a_3+\ .\ .\ .\ +a_n)^2<n(a_1^2+a_2^2+\ .\ .\ .\ +a_n^2)$; and $a_1+a_2+a_3+a_4>4\sqrt[4]{a_1a_2a_3a_4}$.

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10. Which is the greater x^6+y^6 or x^5y+xy^5?

11. If x is real, prove that $x^2-8x+22$ can never be less than 6.

12. If $a_1^2+a_2^2+a_3^2+\ .\ .\ .\ +a_n^2=1=b_1^2+b_2^2+\ .\ .\ .\ +b_n^2$, prove that the expression $a_1b_1+a_2b_2+\ .\ .\ .\ +a_nb_n$ cannot be greater than unity.

13. If $x, y, z, \ .\ .\ .\ a, b, c, \ .\ .\ .$ are positive numbers, prove that $\left(\dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c}\right)\left(\dfrac{a}{x}+\dfrac{b}{y}+\dfrac{c}{z}\right)$ cannot be less than 9, and, generally, that $\left(\dfrac{x}{a}+\dfrac{y}{b}+\ .\ .\ .\ \text{to } n \text{ terms}\right)\left(\dfrac{a}{x}+\dfrac{b}{y}+\dfrac{c}{z}+\ .\ .\ .\ \text{to } n \text{ terms}\right)$ cannot be less than n^2.

14. Prove that, if x, y, z are real numbers,

$$x^2(x-y)(x-z)+y^2(y-z)(y-x)+z^2(z-x)(z-y)$$

is necessarily positive.

15. Prove that $x+\frac{1}{nx}>1+\frac{1}{n}$, if $x>1$ or $<\frac{1}{n}$, n not being less than 1, and x being positive.

16. If $2x-1>10-5x$, find a limit to the value of x.

17. If a, b, c are any real positive numbers, prove that $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$ is not greater than $\frac{a^2+b^2+c^2}{abc}$, and not less than $\frac{\sqrt{a}+\sqrt{b}+\sqrt{c}}{\sqrt{abc}}$.

18. Prove that $(x^2+y^2)^2>4xy(x^2-xy+y^2)$.

19. Determine in what cases $x+\frac{3}{x}>$, or $<$, 4; and find the least value of $\frac{(1+x)(2+x)}{3+x}$.

20. If x and n are both positive, and n integral and $x>1$, prove that $x^n-1>n(x^{\frac{n+1}{2}}-x^{\frac{n-1}{2}})$.

21. Show that $3^n(n+3)^n>2^{2n-1}\,\underline{|n+2}$.

22. Prove that, generally,

$$(a_1^2+a_2^2+\ .\ .\ .\ +a_n^2)(b_1^2+b_2^2+\ .\ .\ .\ +b_n^2) \text{ is less than } (a_1b_1+a_2b_2+\ .\ .\ .\ +a_nb_n)^2.$$

23. Show that $\frac{1.3.5\ .\ .\ .\ (2n-1)}{2.4.6\ .\ .\ .\ 2n}$ is less than $\frac{1}{\sqrt{2n+1}}$.

IV

On Series.

42. A SERIES is a succession of expressions formed, each, after some particular one, according to one and the same law.

Thus a, ar, ar^2 . . . is a series, the law being, that each expression after the first can be obtained by multiplying the preceding by r.

43. The expressions forming a series may be connected in any way we please.

Thus $a.ar.ar^2.ar^3$ . . ., and $a+ar+ar^2+ar^3+$etc., are both series, the first a series of factors, and the second of terms.

Generally, however, unless the contrary is expressly stated, we shall suppose the expressions to be connected by the sign $+$ or $-$, and each will be called a term.

44. We expressed the law of the series in Art. 42, by stating how each term can be obtained when we know the preceding; but it might also have been given in such a manner, that all we need specially to know, in order to determine a particular term, is its place in the series. The algebraic expression of such a law is called the *general* term of the series. Thus, above, ar^{n-1} is the expression for the term which stands in the nth place, so that by giving n the values 2, 5, 10, for example, we obtain from it immediately the 2d, 5th, and 10th terms. Hence it is called the nth, or general, term of the series. See [Art. 419, 430.]

45. Again, in the series $1+2+3+5+9+17+$etc., the law of formation may be given in two ways, by saying (1) that each term, after the second, can be obtained by multiplying the preceding by 2 and subtracting 1 from this product; or (2) that the general, or nth term, is $2^{n-1}-2^{n-2}+1$, n being >1.

46. A series is often stated by giving a few of the terms at the beginning, from which a law of formation can be easily inferred, but if this cannot be done we must have the law explicitly mentioned also.

47. If a series stops at some one term, it is called a *finite* series.

$$\text{Thus } a^m+ma^{m-1}x+\ldots\frac{m(m-1)\ldots(m-r+1)}{\underline{|r}}a^{m-r}x^r+\text{etc.},$$

is a finite series when m is a positive integer.

If a series does not stop, but is endlessly prolonged, it is called an infinite series.

Thus the above series is infinite when m is other than a positive integer, and the series in Art. 43 are both infinite.

48. We have various ways of expressing series generally; the following are examples:—

$$u_0+u_1+u_2+\ \ldots\ u_{n-1}+u_n+\text{etc.} \qquad (1),$$
$$f(a)+f(a+1)+\ \ldots\ +f(a+n)+\text{etc.} \qquad (2),$$
$$u_0+u_1x+u_2x^2+\text{etc.}+u_{n-1}x^{n-1}+u_nx^n+\text{etc.} \qquad (3).$$

We sometimes also write them thus $\Sigma_0^{\infty}u_n$, $\Sigma_0^{\infty}f(a+n)$, $\Sigma_0^{\infty}u_nx^n$, by which we mean that in the general terms, u_n, $f(a+n)$, u_nx^n, we are to put n successively equal to 0 and all integers from 1 onwards, and connect each term so obtained with the succeeding one by the sign $+$.

The series u_0-u_1x+etc.$+(-1)^{n-1}u_{n-1}x^{n-1}+(-1)^nu_nx^n+$ etc. would be denoted by $\Sigma_0^{\infty}(-1)^nu_nx^n$.

The series (3) is said to proceed according to ascending powers of x.

49. We often denote the sum of the first n terms of a series by S_n, so that the sum of an infinite number of terms starting from the first would be denoted by S_∞.

50. It will be well for the student to practise himself in determining a form for the nth, or general, term from inspecting the first few terms of a series.

Thus in the series

$$7,\ 16,\ 22,\ 26,\ 32,\ 36,\ 42,\ \text{etc.},$$

after the second, each term of an odd rank is obtained by adding 6 to the term immediately preceding, and each term of an even rank by adding 4 to the preceding term.

Hence

$$u_{2m}=16+6+6+\text{etc. to } \overline{m-1} \text{ terms}+4+4+\text{ etc. to } \overline{m-1} \text{ terms}$$
$$=16+6(m-1)+4(m-1)=6+10m=6+5.2m,$$

and

$$u_{2m+1}=16+6+6+\text{etc. to } m \text{ terms }+4+4+\text{etc. to } m-1 \text{ terms}$$
$$=16+6m+4(m-1)=12+10m=7+5(2m+1);$$
$$\therefore\ u_{2m}=6\tfrac{1}{2}-\tfrac{1}{2}(-1)^{2m}+5.2m,$$
$$\text{and } u_{2m+1}=6\tfrac{1}{2}-\tfrac{1}{2}(-1)^{2m+1}+5(2m+1);$$

$\therefore$ the general, or nth, term can be put into the form

$$u_n=6\tfrac{1}{2}-\tfrac{1}{2}(-1)^n+5n.$$

See *De Morgan's Algebra*, Chap. VIII.

EXAMPLES.—III.

Write down the general term of each of the following series.

1. $\dfrac{2}{1.3}+\dfrac{4}{3.5}+\dfrac{6}{5.7}+$ etc.

2. $\dfrac{1}{2}+\dfrac{1.3}{2.4}+\dfrac{1.3.5}{2.4.6}+$ etc.

3. $\dfrac{1.2}{\lfloor 3}+\dfrac{3.2^2}{\lfloor 4}+\dfrac{5.2^3}{\lfloor 5}+\dfrac{7.2^4}{\lfloor 6}+$ etc.

4. $\frac{1}{1.2.3}+\frac{x}{2.3.4}+\frac{x^2}{3.4.5}+$ etc.

5. $\frac{1}{(x+1)(3x+1)}+\frac{1}{(2x+1)(4x+1)}+\frac{1}{(3x+1)(5x+1)}+$ etc.

6. $1-2x-5x^2-8x^3-11x^4-$ etc.

7. $1+3+7+15+31+$ etc.

8. $1+4x+7x^2+10x^3+$ etc.

9. $a-2(a+1)x+3(a+2)x^2-4(a+3)x^3+$ etc.

10. $\frac{1}{6.16}+\frac{1}{8.20}+\frac{1}{10.24}+\frac{1}{12.28}+$ etc.

11. $\frac{3}{5}-\frac{5}{8}+\frac{7}{17}-\frac{9}{44}+\frac{11}{125}-$ etc.

12. $\left(\frac{2^2-1}{2^2+1}\right)^2+\left(\frac{3^2-1}{3^2+1}\right)^2 x+\left(\frac{4^2-1}{4^2+1}\right)^2 x^2+$ etc.

13. $1+n+\frac{n(n+1)}{1.2}+\frac{n(n+1)(n+2)}{\underline{|3}}+$ etc.

14.
$$3^r-r.3^{r-1}+\frac{(r-1)(r-2)}{\underline{|2}}3^{r-2}-\frac{(r-2)(r-3)(r-4)}{\underline{|3}}3^{r-3}+\text{etc.}$$

15. $\frac{1}{3.1}+\frac{1}{5\underline{|3}}+\frac{1}{7\underline{|5}}+$ etc.

16. $1+\frac{1}{\underline{|2}}+\frac{2}{\underline{|3}}+\frac{2^2}{\underline{|4}}+\frac{2^3}{\underline{|5}}+$ etc.

51. Let S_n denote the sum of n consecutive terms of a series, starting from any one we please; then if as $n=\infty$ (Art. 26), the values of S_n have a limit (Art. 29, 31), the series is said to be *convergent;* and if as $n=\infty$, $S_n=\infty$, the series is said to be *divergent;* but if S_n is equal sometimes to one number and sometimes to another, the series is said to be *periodic.*

52. Thus the sum of the first n terms of the series $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\text{etc.}$ is $\dfrac{1-\overline{\tfrac{1}{2}|}^{\,n}}{\tfrac{1}{2}}$, which has 2 for its limit, when n is endlessly increased, this series is therefore convergent.

Also the sum of the first n terms of the series $1+2+4+8+\text{etc.}$, is $\overline{2|}^{\,n-1}$, which is infinite (Art. 26, *Obs.* 2), when n is endlessly increased, and therefore this series is divergent.

Again in the series $1-1+1-1+$ etc., the sum of any even number of terms is 0, whilst the sum of an odd number is 1, hence this series is periodic.

53. When a general symbol of number is involved in the terms of a series, the numerical value of the expression for the sum of n terms may have a limit, when n is endlessly increased, for some values of this symbol, but not for others.

Thus in the series $a+ar+ar^2+\text{etc.}$ $S_n=a\dfrac{1-r^n}{1-r}$.

Hence if r has a positive value less than 1, the numerical values of S_n have the value $\dfrac{a}{1-r}$ for their limit, when n is endlessly increased; but if r is greater than 1, the numerical value of S_n is infinite (Art. 26, *Obs.* 2), when n is endlessly increased.

Rules for determining whether a series is convergent or divergent will be given in Chapter V.

54. In the case of a convergent series we denote the limit of S_n, when $n=\infty$, by S, and we often call it the sum of the series *ad infinitum*, or shortly, the sum of the series.

Let $X_n=S-S_n$, then X_n is the sum of the series, starting from the $(n+1)$th term, and is often called the remainder, or remnant, of the series after n terms.

Obviously $X_n=0$ (Art. 27), when $n=\infty$ (Art. 26).

55. The student must remember that the algebraic *form* of S_n is unaltered when n is infinite. It is only the numerical *values* of S_n which then have a certain limit, if the series is convergent.

For in algebra any one term of an expression is just as important as any other, whatever may be the arithmetical values assigned to the symbols involved.

Thus in the series of Art. 53, S_n always has the form $a\dfrac{1-r^n}{1-r}$ whatever r and n may be, it is only its numerical values which have $\dfrac{a}{1-r}$ for their limit, when $n=\infty$, if $r<1$, or >-1.

56. When a function indicates that an operation has to be performed on an expression involved in it, we frequently find that if we perform the operation, there results an infinite series.

The function is called the *generating function* of the series, and the series is called the *development*, or expansion, of the function.

Thus the function $\dfrac{1}{1-x}$ indicates that 1 is to be divided by $1-x$; if we perform this operation we obtain for quotient the infinite series, $1+x+x^2+$ etc.

57. If P denote the generating function of the series $u_0+u_1+u_2+$ etc., we often meet with the expression

$$P=u_0+u_1+u_2+\text{ etc.};$$

this means, not that P represents algebraically the sum of the series to infinity, but merely that if we perform the operations indicated by P, the result will be a series, such that, if the operation be carried on to any one term of it, the result will agree with $u_0+u_1+u_2+$ etc. to the same term.

Thus we often have

$$\frac{1}{1-x}=1+x+x^2+\dots \qquad . \quad . \quad . \qquad (1),$$

and also $$\frac{1}{1-x}=1+x+x^2+\dots \qquad +x^{n-1}+\frac{x^n}{1-x} \qquad (2).$$

Here in (1) the sign = may be read "*is the generating function of the series;*" whilst in (2) it has its usual meaning, namely, strict algebraic and arithmetical equality, so that if any value be given to x on both sides of it, the results are numerically equal.

58. Generally, after an operation indicated in P has been carried on as far as some one term of the series, we find that there is a difference between P and the sum of the series up to the same term, this difference is called the "*remainder.*"

Thus $P=u_0+u_1+u_2+\ \dots\ +u_{n-1}+R_n$.

Here R_n stands for the remainder after n terms, and is always a function of n. Thus in (2) $R_n=\frac{x^n}{1-x}$, and generally $P=S_n+R_n$, or $R_n=P-S_n$.

59. *Ex.* If we carry on for n steps the division indicated by $\frac{a_0+a_1x}{b_0+b_1x+b_2x^2}$, we shall obtain a quotient of the form $c_0+c_1x+\ \dots\ +c_{n-1}x^{n-1}$ and a remainder Ax^n+Bx^{n+1}.

Hence

$$\frac{a_0+a_1x}{b_0+b_1x+b_2x^2}=c_0+c_1x+\dots+c_{n-1}x^{n-1}+\frac{Ax^n+Bx^{n+1}}{b_0+b_1x+b_2x^2} \quad (1).$$

And this form holds good, however large n may be.

We will now show how $c_0, c_1, \dots c_{n-1}$ may be calculated. Since the right-hand side of (1) represents the quotient when (a_0+a_1x) is divided by $b_0+b_1x+b_2x^2$, the product of it and the divisor $b_0+b_1x+b_2x^2$, by [Art. 72], must be identical with a_0+a_1x.

Hence obtaining this product, and arranging it according to powers of x, we see that

$$c_0b_0+x(c_1b_0+c_0b_1)+\ \dots\ x^r(b_0c_r+b_1c_{r-1}+b_2c_{r-2})+\ \dots$$
$$+x^n(A+b_1c_{n-1}+b_2c_{n-2})+x^{n+1}(B+c_{n-1}b_2)$$

must be identical with a_0+a_1x;

$$
\begin{array}{rlr}
\therefore\ c_0 b_0 = a_0, & \quad . \quad . \quad . & (1), \\
c_1 b_0 + c_0 b_1 = a_1, & \quad . \quad . \quad . & (2), \\
\text{etc.} \quad = \text{etc.} & & \\
b_0 c_r + b_1 c_{r-1} + b_2 c_{r-2} = 0, & \quad . \quad . \quad . & (r+1), \\
\text{etc.} \quad = 0, & & \\
A + b_1 c_{n-1} + b_2 c_{n-2} = 0, & \quad . \quad . \quad . & (n+1), \\
B + c_{n-1} b_2 = 0, & \quad . \quad . \quad . & (n+2).
\end{array}
$$

These equations give us means for obtaining in succession the coefficients $c_0, c_1, \ldots c_{n-1}$, and A and B.

60. It may be observed that the coefficient of any one term, such as c_r, may be obtained from equation $(r+1)$ as soon as the coefficients of the two preceding terms are known. Thus the law of the series is known.

61. Here we have been able to show how to calculate the successive terms of the development, and also the remainder after any given number of terms.

In Algebra, however, we can often find the generating function of a series, or the development of a function, without being able to find the *remainder;* but in the Differential Calculus we have theorems which enable us to expand such functions as we are generally concerned with, and to obtain as well the form of the remainder after any number of terms.

62. If after some finite number of terms R_n vanishes, the development stops at that point, and P has been expanded into a finite series, for the sum of which it is also the algebraic expression.

63. The student must distinguish between the meanings of the expressions represented in this Chapter by R_n and X_n.

We have $S_n = P - R_n$. Hence, if the numerical values of R_n have a limit, say R, when n is infinite, S_n has also a limit, viz., $P-R$, and the series is convergent, and $S=P-R$.

If, moreover, as a particular case of the above, when n is

infinite, $R_n=0$ (Art. 27), then P is the limit of S_n, or the generating function is the sum of the series *ad infinitum*, and in this case only $P=S$.

64. Generally it is found by means of theorems in the Differential Calculus, that if the series is convergent, $R_n=0$ when n is infinite, and in Algebra we always *assume* that it is so, and therefore also that the generating function of a convergent series is its sum *ad infinitum*.

Thus in Euler's Proof of the Binomial Theorem [Art. 425], we show that $(1+x)^m$ is the generating function of the series $1+mx+\frac{m(m-1)}{1.2}x^2+$ etc., and then for all values of x and m which make the series convergent we always assume that $(1+x)^m$ is the sum to infinity. This assumption is shown to be correct by one of the theorems in the Differential Calculus above referred to.

65. Thus to sum up, the generating function can never express the algebraic *form* of the sum to infinity. For however many terms we take into the sum, there is always a difference between its form and that of the generating function. But in most cases of convergent series the value of the expression for this difference endlessly decreases as we increase the number of terms taken into the sum; and therefore the numerical value of the generating function is the limit of the sum to infinity.

V

Convergence and Divergence.

66. In Art. 52, 53, in order to discover whether the series were divergent or convergent, we discussed the expression for the sum of n terms.

There are, however, many series for which we are unable to obtain the corresponding expression.

We shall, therefore, in this Chapter consider some other ways of determining the convergence or divergence of a given series.

67. First, it is obvious that a series is divergent, if from and after some definite term each is equal to, or greater than, some finite number.

For let $u_0+u_1+u_2+$etc. denote the series, and S_r the sum of the first r terms.

1°. Let $u_r=u_{r+1}=$etc.$=$ some finite number β.

Then $S_{r+m}=S_r+m\beta$, and therefore, when $m=\infty$ (Art. 26), $S_{r+m}=\infty$. Hence the series is divergent (Art. 51).

2°. If from and after the rth term each is greater than some finite number β, then each term is greater than the corresponding term of 1°; hence S_{r+m} is greater than in 1°, and therefore, *à fortiori*, the series is divergent.

Cor. A series is divergent if from and after some term (say the rth) each is equal to, or greater than, the preceding.

For then each term after the rth is equal to, or greater than, the finite number u_r, and therefore the series is divergent. This is sometimes expressed by saying that a series is divergent, if from and after some definite term the ratio of each to the preceding is equal to, or greater than, unity.

We have now, therefore, only to discuss those series in which, from and after some definite term each is less than the preceding, or, in other words, the ratio of each to the preceding is less than unity.

68. The first method we shall employ will be best explained by some examples of it.

To show that the series

$$1+1+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}+\text{etc.}, \qquad . \quad . \qquad (1),$$

is convergent.

$$\text{Since } \frac{1}{\lfloor 3}<\frac{1}{2^2}, \ \frac{1}{\lfloor 4}<\frac{1}{2^3}, \ \frac{1}{\lfloor 5}<\frac{1}{2^4}, \text{ etc.,}$$

each term of (1) after the third is less than the corresponding term of the series,

$$1+1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\text{ etc.};$$

∴ the sum of an infinite number of its terms, beginning with the first, is

$$<1+\frac{1}{1-\frac{1}{2}}<3;$$

∴ the given series is convergent (Art. 51).

69. The following is an important example :—

If n *be a positive number the series*

$$1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\text{etc.}$$

is convergent, or divergent, according as n *is, or is not, greater than unity.*

I. Let $n>1$.

The second and third terms are together $<\frac{2}{2^n}<\frac{1}{2^{n-1}}$; the next four terms $\frac{1}{4^n}, \frac{1}{5^n}, \frac{1}{6^n}, \frac{1}{7^n}$ are together $<\frac{4}{4^n}<\frac{1}{2^{2n-2}}$

the next eight terms are together less than $\frac{8}{8^n} < \frac{1}{2^{3n-3}}$; after this taking 2^4 terms together, then 2^5, and so on, we see that the sum of the series *ad infinitum* is less than

$$1+\frac{1}{2^{n-1}}+\frac{1}{2^{2n-2}}+\text{ etc.},$$

and $\therefore$ less than $\dfrac{1}{1-\dfrac{1}{2^{n-1}}}$, which $=\dfrac{2^{n-1}}{2^{n-1}-1}=1+\dfrac{1}{2^{n-1}-1}$;

$\therefore$ the series is convergent.

II. Let $n=1$.

The third and fourth terms are together $> \frac{2}{4}$ which $=\frac{1}{2}$; the next four terms, $\frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}$, are together $> \frac{4}{8}$ which $=\frac{1}{2}$; the next eight are together $> \frac{8}{16}$, and so on.

Hence the sum of the series *ad infinitum* is greater than $1+\frac{1}{2}+\frac{1}{2}+$ etc., and therefore goes on endlessly increasing, the series therefore is divergent.

III. Let $n < 1$.

Each term after the first is greater than the corresponding term of the series considered in II., and therefore, *à fortiori*, the series is divergent.

70. Upon the method of proof pursued in the preceding articles we will make a few remarks.

In Article 68 we have compared each term of the series

$$1+\frac{1}{2}+\frac{1}{\underline{|3}}+\text{ etc.}$$

with the corresponding term of the geometric series

$$1+\frac{1}{2}+\frac{1}{2^2}+\text{ etc.},$$

and have shown that each term after the second is less than the corresponding term of the G.P., which we know to be convergent. Hence we conclude that the given series is convergent.

In Article 69, I., we have cut up the given series into batches, the first term forming the first batch, the next two terms forming the second batch, the next four the third, and so on, thus forming a new series by taking each of these batches as a term, at the same time showing that each term of this new series, after the first, is *less* than the corresponding term of the G.P.,

$$1+\frac{1}{2^{n-1}}+\frac{1}{2^{2n-2}}+\text{ etc.},$$

which we know to be convergent. Hence we conclude that the given series is convergent.

Again in Art. 69, II., the new series which we form by taking the terms of the given series in batches, is shown to be, term for term after the second, greater than a series which we know to be divergent, and therefore the given series is seen to be divergent.

Hence we arrive at the following rule to show that a series is convergent [*or* divergent]. Transform it, when necessary, into a new series, convenient for comparing with a known convergent [*or* divergent] series; if after any fixed term each term of the new series is less [*or* greater] than the corresponding term of the known series, then the given series is convergent [*or* divergent].

EXAMPLES.—IV.

1. Prove that the series $\frac{1}{\sqrt{1.2}}+\frac{1}{\sqrt{2.3}}+\frac{1}{\sqrt{3.4}}+$ etc. is divergent, and that $\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+$etc. is divergent.

Are the following seven series convergent or divergent?

2. $1+\frac{1+2}{1+2^2}+\frac{1+3}{1+3^2}+\frac{1+4}{1+4^2}+$etc.

3. The series whose nth term is $\sqrt{n^2+1}-n$.

4. $\dfrac{1}{2}+\dfrac{1}{1+\sqrt{2}}+\dfrac{1}{1+\sqrt{3}}+\dfrac{1}{1+\sqrt{4}}+$ etc.

5. The series whose nth term is $\dfrac{n^3+1}{n^4+1}$.

6. $\dfrac{1}{\sqrt{2}-1}+\dfrac{1}{\sqrt{3}-1}+\dfrac{1}{\sqrt{4}-1}+$ etc.

7. $1+\dfrac{1}{\sqrt{2}}+\dfrac{1}{\sqrt{3}}+\dfrac{1}{\sqrt{4}}+$ etc.

8. $\dfrac{1}{\sqrt{2}}+\dfrac{1}{\sqrt{5}}+$ etc. $+\dfrac{1}{\sqrt{n^2+1}}+$ etc.

9. Show that the series

$$2+\frac{3}{2^m}+\frac{4}{3^m}+\frac{5}{4^m}+\text{ etc.}$$

is convergent, if $m>2$, and divergent if $m=$, or $<$, 2.

10. The series $u_0+u_1+u_2+$ etc. is convergent, if from and after some definite term $(u_n)^{\frac{1}{n}}$ is <1, and divergent, if otherwise.

11. The series $u_0+u_1+u_2+$ etc. is divergent, if the limit of nu_n is not zero when n is infinite.

12. If $n^k u_n$, where k is greater than 1, is always finite however large n may be, then the series, whose nth term is u_n, is convergent.

13. If the terms of the series $u_0+u_1+u_2+$ etc. constantly decrease, then it, and the series $u_0+2u_1+4u_3+8u_7+16u_{15}+$etc. are both convergent, or both divergent.

71. The foregoing method is useful when (but not unless), we can remember some series with which we may conveniently compare the given series.

The following method will be more immediately applicable in many cases.

PROP. *The series* $u_1+u_2+\ldots+u_r+$etc. *is convergent, if after some fixed term* (u_r) *the ratio of each term to the preceding is less than a number which is itself less than unity.*

Let x be this number, S_{r-1} the sum of the first $\overline{r-1}$ terms.

Then $u_{r+1}<xu_r$, $u_{r+2}<xu_{r+1}<x^2u_r$, $u_{r+3}<xu_{r+2}<x^3u_r$, and so on;

$\therefore$ the sum of the series $<S_{r-1}+u_r(1+x+x^2\ \ldots\ ad\ inf.)$

$$<S_{r-1}+u_r\frac{1}{1-x},\ \text{since } x<1;$$

$\therefore$ the sum of the series, when the number of terms taken is endlessly increased, does not exceed the finite number $S_{r-1}+\dfrac{u_r}{1-x}$, and therefore the series is convergent.

72. *Ex.* For what values of x is the series $1+11x+111x^2+1111x^3+$ etc. convergent, and for what values is it divergent?

The nth term $=\dfrac{10^n-1}{9}x^{n-1}$.

$(n-1)$th „ $=\dfrac{10^{n-1}-1}{9}x^{n-2}$;

$\therefore$ the ratio $=\dfrac{10^n-1}{10^{n-1}-1}x$.

Hence, if $x=\dfrac{1}{10}$, or $>\dfrac{1}{10}$, this ratio $=$, or $>$, $1+\dfrac{9}{10^n-10}$, which is never less than 1;

$\therefore$, (Art. 67, Cor.), the series is divergent.

If $x<\dfrac{1}{10}$, the ratio is <1, when $(10^n-1)x<10^{n-1}-1$,

$$\text{or } 1-x<10^{n-1}-10^nx,$$

$$\text{„ } 10^{n-1}(1-10x)>1-x,$$

$$\text{„ } 10^{n-1}>\frac{1-x}{1-10x},$$

$$\text{or } n-1>\log_{10}\frac{1-x}{1-10x}.$$

Hence if m be the characteristic of $\log_{10}\frac{1-x}{1-10x}$, when $n=m+1$ the ratio is <1, and since it continually diminishes, for all higher values of n it is less than when $n=m+1$, *i.e.*, it is less than a number which is itself less than unity. Hence (Art. 71) the series is convergent.

73. By Art. 67 all series are divergent in which, from and after some definite term, the ratio of each to the preceding is equal to, or greater than, unity.

By Art. 71 all series are convergent in which, from and after some definite term, the ratio of each to the preceding is less than a number which is itself less than unity.

These two Articles give a complete test except in the case of those series in which, from and after some definite term, the ratio of each to the preceding, though always less than unity, tends to unity as its limit.

For such series we must adopt the method explained in Art. 69, 70, or proceed as in Art. 75.

74. If there be two series

$a_0+a_1+a_2+$ etc. . (1), and $b_0+b_1+b_2+$ etc. . (2),

of which (1) is convergent, and (2) such that, from and after some term (say the rth), the ratio of each term to the preceding is less than the corresponding ratio in (1), then (2) is convergent.

For $(2)=b_0+\ldots+b_{r-1}+b_r\left(1+\frac{b_{r+1}}{b_r}+\frac{b_{r+2}}{b_{r+1}}\frac{b_{r+1}}{b_r}+\text{ etc.}\right)$,

which $<b_0+\ \ldots\ +b_{r-1}+b_r\left(1+\frac{a_{r+1}}{a_r}+\frac{a_{r+2}}{a_{r+1}}\frac{a_{r+1}}{a_r}+\text{etc.}\right)$,

which $=b_0+\ \ldots\ +b_{r-1}+\frac{b_r}{a_r}(a_r+a_{r+1}+a_{r+2}+\text{etc.})$.

And this latter series is convergent, since (1) is; $\therefore$ (2) is also.

COR. If (1) be divergent, and the ratios $\frac{b_{r+1}}{b_r}$, etc. be greater than $\frac{a_{r+1}}{a_r}$, etc., then (2) is divergent.

75. PROP. *Let* $u_0+u_1+u_2+$etc. *be a series in which the ratio* $\frac{u_{n+1}}{u_n}$ *is less than* 1, *but tends to* 1 *for its limit when* n *increases, and let* $\frac{u_{n+1}}{u_n}$ *be put into the form* $\frac{1}{1+a}$. *Then if* na *is never infinite; but, from and after some definite term, is greater than a number which is greater than* 1, *the series is convergent, and if less than* 1, *divergent.*

Since the limit of $\frac{u_{n+1}}{u_n}$ is 1, that of a is 0.

1°. Suppose that from and after the rth term na is $>\frac{p}{q}$, and that $\frac{p}{q}>1$, so that $p>q$.

Then $(1+a)^q>\left(1+\frac{1}{n}\right)^p$,

$$\text{if } qa+\frac{q(q-1)}{1.2}a^2+\text{ etc. } >\frac{p}{n}+\frac{p(p-1)}{1.2}\frac{1}{n^2}+\text{ etc.,}$$

$$i.e., \text{ if } na+na\frac{q-1}{2}a+\text{ etc. } >\frac{p}{q}+\frac{p}{q}\frac{p-1}{2}\frac{1}{n}+\text{ etc.,}$$

$$i.e. \text{ if } na-\frac{p}{q}>\frac{p}{q}\left(\frac{p-1}{2}\cdot\frac{1}{n}+\text{ etc.}\right)-na\left(\frac{q-1}{2}a+\text{ etc.}\right).$$

Now, from and after the rth term, $na>\frac{p}{q}$. Also, since as n increases a decreases, we can give to n so large a value (say m), that, for m and all higher values of n, $\frac{p-1}{2}\frac{1}{n}+$ etc., and $\frac{q-1}{2}a+$ etc. are so small that this inequality holds good.

Hence from and after the $(m+1)$th term

$$(1+a)^q > \left(1+\frac{1}{n}\right)^p,$$

$$\text{or } \frac{1}{1+a} < \left(\frac{1}{1+\frac{1}{n}}\right)^{\frac{p}{q}},$$

$$\text{or } \frac{u_{n+1}}{u_n} < \frac{\dfrac{1}{(n+1)^{\frac{p}{q}}}}{\dfrac{1}{n^{\frac{p}{q}}}};$$

but the series $1+\dfrac{1}{2^{\frac{p}{q}}}+\ .\ .\ +\dfrac{1}{n^{\frac{p}{q}}}+$etc. is convergent (Art. 69, I.); therefore (Art. 74) the given series is convergent.

2°. Suppose that from and after the rth term $na<1$, then

$$a<\frac{1}{n}, \text{ and } 1+a<1+\frac{1}{n},$$

$$\text{or } \frac{1}{1+a} > \frac{1}{1+\frac{1}{n}},$$

$$\therefore\ \frac{u_{n+1}}{u_n} > \frac{\dfrac{1}{1+n}}{\dfrac{1}{n}};$$

but the series $1+\frac{1}{2}+\frac{1}{3}+$ etc., is divergent (Art. 69, II.); therefore (Art. 74, Cor.) the given series is divergent.

76. *Ex.* Show that the series

$$1+\frac{1}{2}\cdot\frac{1}{3}+\frac{1.3}{2.4}\cdot\frac{1}{5}+\frac{1.3.5}{2.4.6}\cdot\frac{1}{7}+\text{ etc.}$$

is convergent.

The $(n+1)$th term is $\dfrac{1.3.5\,..\,(2n-3)(2n-1)}{2.4.6\,..\,\;2(n-1).2n}\cdot\dfrac{1}{2n+1}$,

" nth, " $\dfrac{1.3.5\,..\,(2n-3)}{2.4.6\,..\,2(n-1)}\cdot\dfrac{1}{2n-1}$;

$$\therefore \text{ the ratio is } \frac{2n-1}{2n}\cdot\frac{2n-1}{2n+1}=\frac{4n^2-4n+1}{4n^2+2n},$$

and this is less than 1, but tends constantly to 1 as its limit when n increases.

Now this ratio $=\dfrac{1}{1+\dfrac{6n-1}{4n^2-4n+1}}$, so that $a=\dfrac{6n-1}{4n^2-4n+1}$;

$$\therefore na=\frac{6n^2-n}{4n^2-4n+1}=\frac{6-\dfrac{1}{n}}{4-\dfrac{4}{n}+\dfrac{1}{n^2}}$$

which has $\frac{6}{4}$ for its limit when n is infinite, and is always $>\frac{6}{4}$, and this being >1 the series is convergent.

EXAMPLES.—V.

1. Prove that the series $x+\dfrac{x^2}{2}+\dfrac{x^3}{3}+$ etc. is divergent, if $x=1$, or >1, and convergent, if $x<1$.

2. Prove that the series $1+x+\dfrac{x^2}{\lfloor 2}+\dfrac{x^3}{\lfloor 3}+$ etc. is convergent.

Determine whether the following six series are divergent or convergent.

3. $e^{-x}+e^{-x^2}+e^{-x^3}+$ etc.

4. $1+\dfrac{1}{2}x+\dfrac{1.3}{2.4}x^2+\dfrac{1.3.5}{2.4.6}x^3+$ etc.

5. The series whose nth term is $\dfrac{2n^2+3n+1}{(n+1)(n+2)(n+3)}x^n$.

6. The series whose nth term is $\dfrac{n^3}{n^2+1}x^n$.

7. $1+2^2x+3^2x^2+4^2x^3+$ etc.

8. $1-nx+\dfrac{n(n-1)}{1.2}x^2-\dfrac{n(n-1)(n-2)}{1.2.3}x^3+$ etc., x being a positive number.

9. Prove that the series

$$1+x+\frac{x(nx-1)}{\underline{|2}}\frac{1}{n}+\frac{x(nx-1)(nx-2)}{\underline{|3}}\frac{1}{n^2}+\text{ etc.}$$

is convergent, if n is greater than 1.

10. Prove that the series

$$x^m+\frac{m(m-1)}{1.2}x^{m-2}y^2+\frac{m(m-1)(m-2)(m-3)}{\underline{|4}}x^{m-4}y^4+\text{ etc.}$$

is convergent, x being greater than y.

77. We have treated hitherto only of series with all positive terms. If the terms be all negative, the same methods will evidently be applicable, as the divergence or convergence of a series depends only on the magnitude of its sum when carried on *ad infinitum.*

If we have a series with alternately positive and negative terms, there is one case in which we can shew that it is convergent, viz., when each term is less than the preceding.

Let the series be $u_1-u_2+u_3-u_4+$ etc.

Then $u_r<u_{r-1}$, and such a number as $u_{r-1}-u_r$ is positive.

By writing the series thus

$$(u_1-u_2)+(u_3-u_4)+(u_5-u_6)+\text{ etc.,}$$

since each number within brackets is positive, we see that the whole series is positive.

By writing it thus

$$u_1-(u_2-u_3)-(u_4-u_5)-\text{ etc.,}$$

since each number within brackets is positive, we see that the whole series is less than u_1; therefore the series is convergent.

EXAMPLES.—VI.

Prove that the following series are convergent:—

1. $\left(\frac{2+1}{2-1}\right)^{\frac{1}{2}}-\left(\frac{3+1}{3-1}\right)^{\frac{1}{3}}+\ldots+(-1)^{n}\left(\frac{n+1}{n-1}\right)^{\frac{1}{n}}+\text{etc.}$

2. $\frac{1}{1.2}+\frac{1}{3.4}+\frac{1}{4.5}+\text{etc.}$

3. $\frac{1}{2.5}+\frac{1}{8.11}+\frac{1}{14.17}+\frac{1}{20.23}+\text{etc.}$

4. $x-\frac{x^3}{\underline{|3}}+\frac{x^5}{\underline{|5}}-\text{etc.}$

5. $1-\frac{x^2}{\underline{|2}}+\frac{x^4}{\underline{|4}}-\text{etc.}$

6. $\frac{1}{1.3}+\frac{1}{5.7}+\frac{1}{9.11}+\text{etc.}$

7. Under what condition is the series

$$x-\frac{x^3}{1+3r}+\frac{x^5}{1+5r}-\frac{x^7}{1+7r}+\text{etc. convergent?}$$

EXAMPLES.—VII.

1. Find whether the following series are convergent or divergent:—

(1.) $\frac{2}{2^2-1}+\frac{3}{3^2-1}+\frac{4}{4^2-1}+\text{etc.}$

(2.) $\frac{2}{2^2+1}+\frac{3}{3^2+1}+\frac{4}{4^2+1}+\text{etc.}$

2. Under what condition is

$$\frac{a}{m+p}+\frac{a^2}{m+2p}+\frac{a^3}{m+3p}+\text{etc. convergent or divergent?}$$

3. Prove that $1+2^{-\frac{3}{2}}+3^{-\frac{4}{3}}+4^{-\frac{5}{4}}+$ etc. is divergent.

4. Prove that $1+\dfrac{1}{(2\log_e 2)^{\mu}}+\dfrac{1}{(3\log_e 3)^{\mu}}+\dfrac{1}{(4\log_e 4)^{\mu}}+$ etc.,

and $1+\dfrac{1}{(\log_e 2)^{\mu}}+\dfrac{1}{(\log_e 4)^{\mu}}+\dfrac{1}{(\log_e 8)^{\mu}}+\dfrac{1}{(\log_e 16)^{\mu}}+$ etc.,

are convergent, or divergent, according as μ is, or is not, greater than 1.

5. If a be a whole number, then the two series

$u_1+u_2+u_3+$ etc., and $u_1+au_a+a^2u_{a^2}+a^3u_{a^3}+$ etc.,

are both convergent, or both divergent.

6. Prove that $\dfrac{x}{1+x}+\dfrac{2x^2}{(1+x)^2}+\dfrac{3x^3}{(1+x)^3}+$ etc. is convergent when x is positive.

VI

Exponential Series and Logarithmic Series.

78. As we shall use the series

$$1+1+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\text{etc.} \quad . \quad . \quad . \qquad (1),$$

we here make a few remarks upon it.

1°. Its sum, denoted by e in [Art. 464], is greater than 2. For the first two terms amount to 2 and all the other terms are positive.

2°. Again e is less than 3. For $\frac{1}{\underline{|3}}<\frac{1}{2^2}$, $\frac{1}{\underline{|4}}<\frac{1}{2^3}$, etc.;

$\therefore$ the series is less than

$$1+1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\text{etc., } \textit{ad infin.}, <1+\frac{1}{1-\frac{1}{2}}<3.$$

Hence e lies between 2 and 3.

By taking the sum of the first twelve terms of (1) the student will find the value of e (correct to 7 places of decimals) to be 2·7182818. The terms after the twelfth only affect the digits after the seventh place.

3°. Also e is a surd [Art. 289]. For if it is not, it can be represented by a fraction $\frac{m}{n}$, where m and n are positive integers [Art. 289];

$$\therefore \frac{m}{n}=2+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\dots\frac{1}{\underline{|n}}+\frac{1}{\underline{|n+1}}+\frac{1}{\underline{|n+2}}+\text{etc.}$$

Multiply both sides by $\underline{|n}$; since $\underline{|n}$ is exactly divisible by each of the factorials $\underline{|2}$, $\underline{|3}$, . . . $\underline{|n}$, and since $\underline{|n}=n\,\underline{|n-1}$,

$$\therefore\ m\,\underline{|n-1}=\text{an integer}+\frac{1}{n+1}+\frac{1}{(n+1)(n+2)}+\text{etc.}$$

$$\text{Now}\ \frac{1}{(n+1)(n+2)}<\frac{1}{(n+1)^2},$$

$$\frac{1}{(n+1)(n+2)(n+3)}<\frac{1}{(n+1)^3},$$

$$\text{etc.}\qquad<\text{etc.};$$

$\therefore$ the series

$$\frac{1}{n+1}+\frac{1}{(n+1)(n+2)}+\text{etc.}<\frac{1}{n+1}+\frac{1}{(n+1)^2}+\text{etc. } \textit{ad infin.}$$

$$<\frac{\frac{1}{n+1}}{1-\frac{1}{n+1}}<\frac{1}{n},$$

and therefore is a proper fraction; hence we have a proper fraction $=m\,\underline{|n-1}-$an integer

$=$an integer,

which is impossible; $\therefore$ e is a surd.

79. Exponential Theorem. *The series for* a^x, *of ascending powers of* x, *is*

$$1+x\log_e a+\frac{x^2\overline{\log_e a}|^2}{\underline{|2}}+\frac{x^3\overline{\log_e a}|^3}{\underline{|3}}+\text{ etc.}$$

Whatever n may be we have

$$\left\{\left(1+\frac{1}{n}\right)^n\right\}^x=\left(1+\frac{1}{n}\right)^{nx}\qquad .\qquad .\qquad (1).$$

Now

$$\left(1+\frac{1}{n}\right)^{nx}=1+nx\frac{1}{n}+\frac{nx(nx-1)}{\underline{|2}}\frac{1}{n^2}+\frac{nx(nx-1)(nx-2)}{\underline{|3}}\frac{1}{n^3}+\text{etc.}\ (2)$$

$$=1+x+\frac{x}{\underline{|2}}\left(x-\frac{1}{n}\right)+\frac{x}{\underline{|3}}\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)+\text{ etc.}$$

Hence *when* n *is infinite* (Art. 26, Obs. 2),

the limit of $\left(1+\frac{1}{n}\right)^{nx}=1+x+\frac{x^2}{\lfloor 2}+\frac{x^3}{\lfloor 3}+$ etc.,

and *then* also, by putting $x=1$, we see that

the limit of $\left(1+\frac{1}{n}\right)^{n}=1+1+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}+\text{etc.}=e$;

$\therefore$ from (1) $e^x=1+x+\frac{x^2}{\lfloor 2}+\frac{x^3}{\lfloor 3}+$ etc.

And this is true whatever form the index of e may have.

But $\log_e a^x = x\log_e a$; $\therefore$ $a^x=e^{x\log_e a}$ [Art. 447];

$$\therefore\ a^x=1+x\log_e a+\frac{\overline{x^2\log_e a}|^2}{\lfloor 2}+\frac{\overline{x^3\log_e a}|^3}{\lfloor 3}+\text{ etc.}$$

This result is called the Exponential Theorem.

The student will observe that we have taken $\left(1+\frac{1}{n}\right)^{nx}$, the generating function of the series in (2), as the sum of the same series. This we do in accordance with the assumption mentioned in Art. 64, since the series is convergent.

EXAMPLES.—VIII.

1. Prove that $n=1+\log_e n+\frac{\overline{\log_e n}|^2}{\lfloor 2}+\frac{\overline{\log_e n}|^3}{\lfloor 3}+$ etc.

2. Express $n+\frac{1}{n}$ in terms of $\log_e n$.

3. Prove that $\frac{1}{e}=\frac{1}{3.1}+\frac{1}{5\lfloor 3}+\text{etc.}...+\frac{1}{(2n+3)\lfloor 2n+1}+$ etc.

4. Obtain a series for $e-e^{-1}$.

5. Prove that $\frac{(e^2-1)^2}{8e^2}=\frac{1}{\lfloor 2}+\frac{2^2}{\lfloor 4}+$ etc. $+\frac{2^{2n-2}}{\lfloor 2n}+$ etc.

6. Express the sum of the series $\frac{1}{\lfloor 2} + \frac{1}{\lfloor 4} +$etc. in terms of e.

7. Show that $\dfrac{e-1}{e+1} = \dfrac{\frac{1}{\lfloor 2}+\frac{1}{\lfloor 4}+\frac{1}{\lfloor 6}+\text{etc., } ad\ infin.}{1+\frac{1}{\lfloor 3}+\frac{1}{\lfloor 5}+\text{etc., } ad\ infin.}$

8. Prove that $\dfrac{1+\frac{1}{\lfloor 2}+\frac{2}{\lfloor 3}+\frac{2^2}{\lfloor 4}+\frac{2^3}{\lfloor 5}+\text{etc.}}{1+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 4}+\text{etc.}} = \dfrac{e}{2}$

9. Investigate a series for $e^{\sqrt{-1}}+e^{-\sqrt{-1}}$.

10. Sum the series $\frac{1}{1.2}+\frac{1.3}{1.2.3.4}+\frac{1.3.5}{1.2.3.4.5.6}+$etc. to infinity.

11. If x be a positive quantity, prove that $x^{\frac{1}{x}}$ is less than e.

80. PROP. *The series for* $\log_e(1+\mathrm{x})$, *of ascending powers of* x, *is*

$$x-\frac{x^2}{2}+\frac{x^3}{3}+\text{etc.},$$

x *being* <1.

By the Exponential Theorem

$$a^v = 1+v\log_e a+v^2\frac{\overline{\log_e a}|^2}{\lfloor 2}+\frac{v^3\overline{\log_e a}|^3}{\lfloor 3}+\text{ etc.};$$

$$\therefore\ \frac{a^v-1}{v} = \log_e a+v\left\{\frac{\overline{\log_e a}|^2}{\lfloor 2}+\frac{v\overline{\log_e a}|^3}{\lfloor 3}+\text{ etc.}\right\}$$

Hence when $v=0$ (Art. 27) the limit of $\frac{a^v-1}{v}$ is $\log_e a$.

For the series within the brackets is convergent (Ex. V. 2), whatever finite value a may have, and therefore the product of it and v vanishes with v (Art. 30);

$$\therefore\ \log_e(1+x) \text{ is the limit of } \frac{(1+x)^v-1}{v} \text{ when } v=0.$$

Now, when x is <1, the series for $(1+x)^v$, obtained from the Binomial Theorem, is convergent;
$\therefore$ then (Art. 64)

$$\frac{(1+x)^v-1}{v}=\frac{\left(1+vx+\frac{v(v-1)}{\underline{|2}}x^2+\frac{v(v-1)(v-2)}{\underline{|3}}x^3+\text{etc}\right)-1}{v}$$

$$=x+\frac{v-1}{\underline{|2}}x^2+\frac{(v-1)(v-2)}{\underline{|3}}x^3+\text{etc.},$$

and the limit of this, when $v=0$, is

$$x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\text{ etc.};$$

$$\therefore \log_e(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\text{etc., when } x<1.$$

81. We proceed to deduce some formulæ for obtaining the Napierian logarithms of numbers.

We have, when x is <1, $\log_e(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-$ etc. (1);

$$\therefore\text{, writing } -x \text{ for } x,\ \log_e(1-x)=-x-\frac{x^2}{2}-\frac{x^3}{3}-\text{ etc.};$$

$$\therefore\text{, subtracting, } \log_e\frac{1+x}{1-x}=2\left(x+\frac{x^3}{3}+\frac{x^5}{5}+\text{ etc.}\right).$$

Putting $\frac{n+1}{n}$ for $\frac{1+x}{1-x}$, and $\therefore$ $\frac{1}{2n+1}$ for x, we obtain

$$\log_e(1+n)-\log_e n=2\left\{\frac{1}{2n+1}+\frac{1}{3}\left(\frac{1}{2n+1}\right)^3+\frac{1}{5}\left(\frac{1}{2n+1}\right)^5+\text{ etc.}\right\}\ (2).$$

Again, put $\frac{m^2}{m^2-1}$ for $\frac{1+x}{1-x}$, and $\therefore$ $\frac{1}{2m^2-1}$ for x;

$\therefore$ $\log_e m^2-\log_e(m^2-1)$, *i.e.* $2\log_e m-\log_e(m-1)-\log_e(m+1)$

$$=2\left\{\frac{1}{2m^2-1}+\frac{1}{3}\left(\frac{1}{2m^2-1}\right)^3+\frac{1}{5}\left(\frac{1}{2m^2-1}\right)^5+\text{ etc.}\right\}\quad(3).$$

In (2) put $n=1$ and we have, since $\log 1=0$,

$$\log_e 2=2\left\{\frac{1}{3}+\frac{1}{3}\cdot\frac{1}{3^3}+\frac{1}{5}\cdot\frac{1}{3^5}+\text{ etc. }\right\},$$

and by taking a sufficient number of terms of this series, we obtain $\log_e 2$ to any required degree of accuracy.

In (3) put $m=2$ and we have

$$\log_e 3-2\log_e 2=-2\left\{\frac{1}{7}+\frac{1}{3}\,\frac{1}{7^3}+\frac{1}{5}\,\frac{1}{7^5}+\text{ etc. }\right\},$$

from which, since we have obtained $\log_e 2$, we can now obtain $\log_e 3$.

In (3) put $m=3$ and we have

$$\log_e 4=2\log_e 3-\log_e 2-2\left\{\frac{1}{17}+\frac{1}{3}\,\frac{1}{17^3}+\frac{1}{5}\,\frac{1}{17^5}+\text{ etc. }\right\},$$

which gives us $\log_e 4$, since $\log_e 3$, $\log_e 2$ have been already obtained.

And by putting $m=4$, 5, etc. successively, we obtain the logarithms of all succeeding numbers.

82. *Obs.* The series in (1) Art. **81** gives the logarithm of any number without our knowing the logarithm of any other number.

The formula (2) gives the logarithm of one of the numbers o the form n, and $n+1$, *i.e.*, of two consecutive numbers, when the logarithm of the other is known.

The formula (3) gives the logarithm of one of the numbers of the form $m-1$, m, $m+1$, *i.e.*, of three consecutive numbers, when the logarithms of the other two are known.

The reason why it is preferable to use the formula (2) than to calculate the logarithm of each number separately from (1), is that the series in (2) is a much more *rapidly converging* series than that in (1), by which we mean that the terms decrease much more rapidly in the one series than in the other, and therefore, to approximate to the sum of the series in (2), within any

required degree of accuracy, we need take fewer of its terms than we should of the series in (1) to obtain the same degree of accuracy.

For the same reason the formula (3) is preferable to either (2) or (1).

83. In calculating logarithms we need use the above formulæ for primes only, for the logarithm of any other number can be obtained from the logarithms of its prime factors. Thus $\log 12 = \log 3 + 2\log 2$.

84. Having thus obtained the logarithms of numbers in the Napierian system, we can find the logarithms of the same numbers in the common system, by the method explained in [Art. 465].

EXAMPLES.—IX.

1. Prove that

$$\log_e a = a - 1 - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \frac{1}{4}(a-1)^4 + \text{etc.}$$

2. Prove that

$$\log_e a - \log_e x = \frac{a-x}{a} - \frac{1}{2}\left(\frac{a-x}{a}\right)^2 + \frac{1}{3}\left(\frac{a-x}{a}\right)^3 - \text{etc.}$$

3. Show that $\log_e(1-x^2) = -2\left\{\frac{x^2}{2} + \frac{x^4}{4} + \frac{x^6}{6} + \text{etc.}\right\}$.

4. Prove that

$$2\log_e\left(x + \frac{1}{x}\right) = x^2 + \frac{1}{x^2} - \frac{1}{2}\left(x^4 + \frac{1}{x^4}\right) + \frac{1}{3}\left(x^6 + \frac{1}{x^6}\right) - \text{etc.}$$

5. Prove that $\log_e n = n - \frac{1}{n} - \frac{1}{2}\left(n^2 - \frac{1}{n^2}\right) + \frac{1}{3}\left(n^3 - \frac{1}{n^3}\right) - \text{etc.}$

6. Investigate the formula

$$\log_e(x+2) = 2\log_e(x+1) + \log_e(x-2) - 2\log_e(x-1)$$
$$+ 2\left\{\frac{2}{x^3-3x} + \frac{1}{3}\left(\frac{2}{x^3-3x}\right)^3 + \frac{1}{5}\left(\frac{2}{x^3-3x}\right)^5 + \cdots\right\}.$$

7. **Prove that**

$$\log_e(1+x+x^2)=x+\frac{x^2}{2}-\frac{2}{3}x^3+\frac{x^4}{4}+\frac{x^5}{5}-\frac{1}{2}\cdot\frac{2}{3}x^6$$
$$+\frac{x^7}{7}+\frac{x^8}{8}-\frac{1}{3}\cdot\frac{2}{3}x^9+\text{etc.}$$

8. **Prove that** $n\{x-1-\frac{1}{2}(x-1)^2+\frac{1}{3}(x-1)^3-\text{etc.}\}$

$$=\frac{x^n-1}{x^n}+\frac{1}{2}\left(\frac{x^n-1}{x^n}\right)^2+\frac{1}{3}\left(\frac{x^n-1}{x^n}\right)^3+\text{etc.};$$

and that $\log_e\{1+(1+x)+(1+x)^2\}=3\log_e(1+x)-\log_e x$

$$-\left\{\frac{1}{(1+x)^3}+\frac{1}{2}\,\frac{1}{(1+x)^6}+\frac{1}{3}\,\frac{1}{(1+x)^9}+\text{etc.}\right\}.$$

9. Find x from the equation $2^x+2^{x-1}=10$.

10. If α, β be the roots of the equation $ax^2+bx+c=0$, then

$$\log_e(ax^2+bx+c)=\log_e a+2\log_e x-\frac{1}{x}(\alpha+\beta)-\frac{1}{2x^2}(\alpha^2+\beta^2)$$
$$-\frac{1}{3x^3}(\alpha^3+\beta^3)-\text{etc.}$$

11. If $a_0+a_1x+a_2x^2+\text{etc.}$ be the expansion of $\log_e(1+x)^{\frac{1}{1-x}}$, show that $2n(a_{2n-1}-a_{2n})=1$.

12. Show that $\log_a y=\dfrac{y-1-\frac{1}{2}(y-1)^2+\frac{1}{3}(y-1)^3-\text{etc.}}{a-1-\frac{1}{2}(a-1)^2+\frac{1}{3}(a-1)^3-\text{etc.}}$

VII

Indeterminate Coefficients.

85. Prop. *If any positive integral function of* x *of the* n*th degree vanish for more than* n *different values of* x, *then each of the coefficients must vanish.*

Let the function be $Nx^n+Mx^{n-1}+\ldots+Cx^2+Bx+A$. Denote it by $f(x)$ for shortness.

Suppose $f(x)$ vanishes when $x=a_1, a_2, \ldots a_n$ and a_{n+1}.

Then (App. Art. 12), $f(x)=N(x-a_1)(x-a_2)\ldots(x-a_n)$.

Now when $x=a_{n+1}$, $f(x)=0$; but none of the expressions $a_{n+1}-a_1$, $a_{n+1}-a_2, \ldots a_{n+1}-a_n$ vanish, since a_{n+1} is different from each of the symbols $a_1, a_2 \ldots a_n$;

$$\therefore \textit{ then } N=0;$$

but the value of N does not depend on x;

$$\therefore \textit{ always } N=0.$$

Hence $f(x)$ reduces to $Mx^{n-1}+\ldots+Cx^2+Bx+A$, which therefore vanishes when $x=a_1, \ldots a_{n+1}$. Hence we can show as above that M and each of the other coefficients vanish.

We add another proof of this proposition.

86. If the expression $A+Bx+Cx^2+Dx^3=0$ for more than three values of x, then $A=0$, $B=0$, $C=0$, $D=0$.

For let α, β, γ, δ be four values of x such that

$$A+B\alpha+C\alpha^2+D\alpha^3=0 \quad . \quad . \quad (1),$$

$$A+B\beta+C\beta^2+D\beta^3=0 \quad . \quad . \quad (2),$$

$$A+B\gamma+C\gamma^2+D\gamma^3=0 \quad . \quad . \quad (3),$$

$$A+B\delta+C\delta^2+D\delta^3=0 \quad . \quad . \quad (4).$$

From (1) and (2) by subtraction we have

$$(\alpha-\beta)\{B+C(\alpha+\beta)+D(\alpha^2+\alpha\beta+\beta^2)\}=0.$$

Now since $\alpha-\beta$ is not $=0$, α and β being different;

$$\therefore B+C(\alpha+\beta)+D(\alpha^2+\alpha\beta+\beta^2)=0.$$

Similarly $\quad B+C(\alpha+\gamma)+D(\alpha^2+\alpha\gamma+\gamma^2)=0 \quad . \quad (5);$

$\therefore$, subtracting, $\quad (\beta-\gamma)\{C+D\alpha+\beta+\gamma\}=0;$

but $\quad \beta-\gamma$ is not $=0$;

$$\therefore C+D(\alpha+\beta+\gamma)=0.$$

Similarly $\quad C+D(\alpha+\beta+\delta)=0; \quad . \quad . \quad (6);$

$\therefore$, subtracting, $\quad (\gamma-\delta)D=0;$

but $\quad \gamma-\delta$ is not $=0$;

$$\therefore D=0;$$

$\therefore$ from (6) $C=0$, and $\therefore$ from (5) $B=0$, and $\therefore$ from (1) $A=0$.

In the same way it can be shown that, *if any positive integral function of* x *of the* n*th degree vanish for more than* n *values of* x, *then each of the coefficients must vanish.*

87. Cor. 1. *If* $\mathrm{A}+\mathrm{B}x+\ldots+\mathrm{N}x^n=\mathrm{A}'+\mathrm{B}'x+\ldots+\mathrm{N}'x^n$ *for more than* n *values of* x, *then* $\mathrm{A}=\mathrm{A}'$, $\mathrm{B}=\mathrm{B}'$, etc., $\mathrm{N}=\mathrm{N}'$.

For since $(A-A')+(B-B')x+\ldots+(N-N')x^n=0$, for more than n values of x; $\therefore A-A'=0$, etc., $N-N'=0$.

88. From this Proposition we cannot assume, if an infinite series involving positive integral powers of x vanishes for an infinite number of values of x, that therefore each of the coefficients vanishes. For being an infinite series it is not of any definite degree.

To this point we shall recur in Art. 93.

89. *Ex.* Given $A+Bx+Cx^2+Dx^3+Ex^4=1+5x^2-9x^4$ for all (or at any rate for more than four) values of x, then $A=1$, $B=0$, $C=5$, $D=0$, $E=-9$.

90. The statement in this corollary is called the Principle of Indeterminate Coefficients, these two words meaning not "coefficients which cannot be determined," but "coefficients which have to be determined."

By this principle, if two positive integral functions of x are equal to one another for a number of values greater than the degrees of the functions, we are allowed to *equate the coefficients* of the like powers of x on each side of the equation.

91. Cor. 2. If two positive integral functions of x be equal for a number of values of x greater than the degrees of the functions, the highest powers of x in both functions must be the same.

For if

$a_nx^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0=b_{n+1}x^{n+1}+b_nx^n+\ldots+b_0,$
for a number of values of x greater than $n+1$, then $b_{n+1}=0$.

92. *Ex.* 1. Investigate the relation which exists between m and n when $mx^3-(2m^2+3n)x^2+(m^3+6mn)x-3m^2n$ is a perfect cube.

When it is a perfect cube it must be of the form

$$a^3x^3+3a^2bx^2+3ab^2x+b^3;$$

$\therefore$, equating coefficients, we have

$$a^3=m, \quad \ldots \quad (1),$$
$$3a^2b=-(2m^2+3n), \quad \ldots \quad (2),$$
$$3ab^2=m^3+6mn, \quad \ldots \quad (3),$$
$$b^3=-3m^2n \quad \ldots \quad (4).$$

From these equations we now eliminate a and b.

By (1) and (4) $\quad a^3b^3=-3m^3n,$

" (2) and (3) $\quad 9a^3b^3=-(2m^2+3n)(m^3+6mn);$

$$\therefore 27m^2n=(2m^2+3n)(m^2+6n)$$
$$=2m^4+15nm^2+18n^2,$$
$$\text{or } m^4-6m^2n+9n^2=0;$$
$$\therefore m^2-3n=0.$$

Ex. 2. Determine the relation which exists between p and q when x^3+px+q is divisible by a factor of the form $(x-a)^2$.

Let $x+b$ be the other factor;

$$\therefore x^3+px+q=(x+b)(x-a)^2$$
$$=x^3+x^2(b-2a)+x(a^2-2ab)+a^2b;$$
$$\therefore b-2a=0, \text{ and } \therefore \quad b=2a;$$
$$a^2-2ab=p, \text{ and } \therefore -3a^2=p;$$
$$a^2b=q, \text{ and } \therefore \quad 2a^3=q;$$
$$\therefore \left(-\frac{p}{3}\right)^3=a^6=\left(\frac{q}{2}\right)^2;$$
$$\therefore 27q^2+4p^3=0.$$

EXAMPLES.—X.

1. Determine the relations which exist amongst a, b, c, d, e, p, q, when $ax^4+bx^3+cx^2+dx+e$ is divisible by the factor x^2+px+q.

2. Investigate the condition for the expression
$$4x^4-4px^3+4qx^2+2p(m+1)x+(m+1)^2$$
being a perfect square.

3. Investigate the condition for the expression
$$Ax^2+2Bxy+Cy^2+2Dx+2Ey+F$$
being divisible by a factor of the form $ax+by+c$.

4. Resolve $2x^2-21xy-11y^2-x+34y-3$ into its factors.

5. Express $4(x^4+x^3+x^2+x+1)$ as the difference between two squares.

6. Investigate the relations between the coefficients that the equation $ax^3+bx^2+cx+d=0$ may be put under one of the forms—

(1) $x^4=(x^2+px+q)^2$,

(2) $q^2=(x^2+px+q)^2$.

Solve in this way the equation, $2x^3-x^2-2x+1=0$.

7. If the two expressions x^3+px^2+qx+r, $x^3+p'x^2+q'x+r'$, have the same quadratic factor, then $\frac{r-r'}{p-p'}=\frac{p'r-pr'}{q-q'}=\frac{q'r-qr'}{r-r'}$.

Show also that the third factors are $x+\frac{p-p'}{r-r'}r$ and $x+\frac{p-p'}{r-r'}r'$;

and that the quadratic factor is $x^2+\frac{q-q'}{p-p'}x+\frac{r-r'}{p-p'}$.

93. Prop. *If the series* $a_0+a_1x+a_2x^2+a_3x^3+$ etc. *is convergent, and equal to zero for all values of* x *between certain limits including zero, then all the coefficients* a_0, a_1, etc. *vanish.*

For all these values of x the series $a_1+a_2x+a_3x^2+$etc. is convergent; for in it the test ratio is the same as in the given series. Hence its sum is equal to zero, or some finite number. Denote it by L;

$$\therefore\ a_0+xL=0.$$

Now L not being infinite, when $x=0$, $xL=0$ (Art. 30); $\therefore a_0=0$;

$$\therefore\ xL=0 \text{ always};$$

$\therefore$ either $x=0$, or, $L=0$ [Art. 324], so that we are certain that L vanishes for all the values of x which we are considering, except when $x=0$, and it *may* also vanish then; and since it vanishes for all the other values of x which we are considering, *however small*, we *infer* that it *does* vanish when $x=0$.

Therefore again we have a series $a_1+a_2x+a_3x^2+$ etc., which is convergent, and equal to zero for all values of x between certain limits including zero;

$\therefore$, in the same way as above, we can show that $a_1=0$; and we can go on to show that each of the other coefficients, in succession, vanishes.

Cor. If for all values of x between certain limits, including zero, the expressions $a_0+a_1x+a_2x^2+$ etc., $b_0+b_1x+b_2x^2+$ etc. are, one a convergent series, and the other a finite expression

or a convergent series, and are also equal to one another; then the coefficients of any the same power of x in both expressions are equal to one another.

For their difference, $a_0-b_0+(a_1-b_1)x+(a_2-b_2)x^2+$etc., is a convergent series which vanishes for these values of x,

$$\therefore a_0-b_0,\ a_1-b_1, \text{ etc., all vanish;}$$
$$\therefore a_0=b_0,\ a_1=b_1, \text{ etc.}$$

94. Prop. *A function of any symbol* x, *which has only one form, can only be expanded into one convergent series of ascending powers of* x.

For if possible let it be expanded into two series, say

$$a_0+a_1x+a_2x^2+ \text{ etc., and } b_0+b_1x+ \text{ etc.}$$

Then since both these series are, by hypothesis, convergent, the function, by the assumption of Art. 64, expresses the sum of each to infinity; therefore the two series are equal to one another for all those values of x for which both are convergent;

$$\therefore a_0=b_0,\ a_1=b_1, \text{ etc., } i.e. \text{ the series are identical.}$$

Obs. If a function can assume more than one form, then it can be expanded into one such series corresponding to each form.

Thus the square root of $1+x$ has two forms, and therefore two corresponding series, namely

$$1+\frac{1}{2}x-\frac{1}{8}x^2+ \text{ etc., and } -1-\frac{1}{2}x+\frac{1}{8}x^2- \text{ etc.}$$

95. *Ex.* I. To expand $e^x(1-x)$ in a series of ascending powers of x.

Put $A_0+A_1x+A_2x^2+$ etc. $=\left(1+x+\frac{x^2}{1.2}+\text{ etc.,}\right)(1-x)$.

$$=1-x^2+\text{etc. }+\left(\frac{1}{\underline{|n}}-\frac{1}{\underline{|n-1}}\right)x^n+\text{ etc.}$$

$\therefore$, equating coefficients, $A_n=\frac{1}{\underline{|n}}-\frac{1}{\underline{|n-1}}=\frac{1-n}{\underline{|n}}=-\frac{1}{n}\frac{1}{\underline{|n-2}}$,

$$\text{and } A_0=1,\ A_1=0\,;$$

$$\therefore e^x(1-x)=1-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4\underline{|2}}-\frac{x^5}{5\underline{|3}}-\text{etc.}$$

Ex. 2. Prove that the coefficient of x^r in the expansion of $(1-3x+2x^2)^{-1}$ is $2^{r+1}-1$.

Put $$\frac{1}{1-3x+2x^2}=A_0+A_1x+A_2x^2+\text{ etc.};$$

$$\therefore\ 1=A_0+(A_1-3A_0)x+(2A_0-3A_1+A_2)x^2+\text{ etc.};$$

$\therefore$, equating coefficients, we have $A_n-3A_{n-1}+2A_{n-2}=0$.

Let
$$\begin{aligned} A_{n-2}&=2^{n-1}-1\\ A_{n-1}&=2^n-1\,;\\ \therefore\ A_n&=3(2^n-1)-2(2^{n-1}-1)\\ &=3.2^n-3-2^n+2\\ &=2^n(3-1)-1=2^{n+1}-1. \end{aligned}$$

Hence if A_{n-2} and A_{n-1} have the given form, A_n has also.

But $$A_0=1=2^{0+1}-1,$$

$$\text{and } A_1-3A_0=0\,;\ \therefore\ A_1=3=2^{1+1}-1\,;$$

$\therefore$ A_0 and A_1 have the given form; $\therefore$ A_2, and each of the succeeding coefficients, has also, *i.e.* $A_r=2^{r+1}-1$.

Ex. 3. If $x=a_1y+a_2y^2+a_3y^3+$ etc., required an expression for y in terms of x.

Suppose $$y=b_1x+b_2x^2+b_3x^3+\text{etc.}$$

$$\therefore\ x=a_1b_1x+(a_1b_2+a_2b_1^2)x^2+(a_3b_1^3+2a_2b_1b_2+a_1b_3)x^3+\text{etc.};$$

$$\begin{aligned} \therefore\ a_1b_1&=1,\\ a_1b_2+a_2b_1^2&=0,\\ a_3b_1^3+2a_2b_1b_2+a_1b_3&=0,\\ \text{etc.}\quad&=0. \end{aligned}$$

Thus we obtain sufficient equations to determine the coefficients in turn.

This determination of y in terms of x is called *reversing* the given series.

Ex. 4. To prove that

$$n^n - n(n-1)^n + \frac{n(n-1)}{1.2}(n-2)^n - \text{etc.} = \underline{|n}$$

By the Exponential theorem we have

$$(e^x - 1)^n = (x + \tfrac{1}{2}x^2 + \text{etc.})^n = x^n + \text{ higher powers of } x \quad (1);$$

$$\text{but also } (e^x-1)^n = e^{nx} - ne^{(n-1)x} + \frac{n(n-1)}{1.2}e^{(n-2)x} - \text{etc.} \quad (2).$$

Now, expanding each of the powers e^{nx}, $e^{(n-1)x}$, etc. by the exponential theorem, and picking out the term in each expansion which contains x^n, we see that the coefficient of x^n in (2) is $\frac{n^n}{\underline{|n}} - n\frac{(n-1)^n}{\underline{|n}} + \frac{n(n-1)}{1.2}\,\frac{(n-2)^n}{\underline{|n}} - \text{etc.}$ The coefficient of x^n in (1) is 1. Hence, the series in (1) and that obtained from (2) being both convergent, and the expansions of the same generating function in ascending powers of x, we have, equating these coefficients of x^n,

$$n^n - n(n-1)^n + \frac{n(n-1)}{1.2}(n-2)^n - \text{etc.} = \underline{|n}$$

Obs. The coefficient of x^r in the series obtained from (2) is

$$\frac{n^r}{\underline{|r}} - n\frac{(n-1)^r}{\underline{|r}} + \frac{n(n-1)}{\underline{|2}}\cdot\frac{(n-2)^r}{\underline{|n}} - \text{etc.} \quad . \quad (3);$$

but there is no term in (1) containing a power of x lower than x^n, hence if $r < n$, $n^r - n(n-1)^r + \frac{n(n-1)}{\underline{|2}}(n-2)^r - \text{etc.} = 0.$

Also by taking $r > n$, and finding the coefficient of x^r from (1), and equating to it the expression (3), we obtain another similar theorem for each value of r.

EXAMPLES.—XI.

1. If A_n is the coefficient of x^n in the expansion of $\frac{a+bx}{1+x+x^2}$, then $A_n = A_{n-3}$.

Write down the first six terms of the expansion.

2. If A_n is the coefficient of x^n in the expansion of

$$\frac{e^x}{1-x}, \text{ then } A_n - A_{n-1} = \frac{1}{\underline{|n}}.$$

3. If $y = x - \frac{x^2}{2} + \frac{x^3}{4} -$ etc., find x in a series of ascending powers of y.

4. Expand $\log_e(1+x)^{1-x}$ in a series of ascending powers of x, when x is less than 1.

5. Show, by equating the coefficients of x^n in the expansions of $2\log_e(1-x)$ and $\log_e(1-2x+x^2)$, that

$$2^n - n2^{n-2} + \frac{n(n-3)}{1.2}2^{n-4} - \frac{n(n-4)(n-5)}{1.2.3}2^{n-6} + \text{etc.} = 2.$$

6. If $x - 3y + y^3 = 0$, find y in a series of ascending powers of x.

7. Show that

$$n^{n+2} - n(n-1)^{n+2} + \frac{n(n-1)}{1.2}(n-2)^{n+2} - \text{etc.} = \frac{n(3n+1)\underline{|n+2}}{\underline{|4}}$$

VIII

Binomial Theorem.

96. We shall, in this chapter, give some additional Theorems, and some examples of various problems, connected with the Binomial Theorem.

97. *To find the numerically greatest term in the expansion of* $(\mathrm{a}+\mathrm{x})^{\mathrm{n}}$ *in ascending powers of* x.

The magnitude of any term does not depend on the signs of a and x; we shall therefore consider them both as positive.

The $(r+1)^{\text{th}}$ term can be obtained by multiplying the r^{th} by $\frac{n+1-r}{r}\frac{x}{a}$, the numerical value of which we therefore shall call the multiplier *for* the $(r+1)^{\text{th}}$ term.

If for several consecutive terms the multiplier >1, each of them is greater than the preceding term, and if then the multiplier for some term becomes <1, the terms then begin to decrease, so that just before this term we shall find the greatest of the terms considered. See [Arts. 410, 411, 421].

I. Let n be positive.

Let p be such that $\left(\frac{n+1}{p}-1\right)\frac{x}{a}=1$, or $p=\frac{(n+1)x}{a+x}$, and, if p is not integral, let q be its integral part.

1°. Suppose n is an integer. Then the series terminates after the $(n+1)^{\text{th}}$ term, and r cannot be $>n$.

Therefore $\left(\frac{n+1}{r}-1\right)\frac{x}{a}$ is always positive. It also decreases as r increases; $\therefore$ it >1 when $r<p$, and <1 when $r>p$.

Hence, if p is an integer, the p^{th} and the $(p+1)^{\text{th}}$ terms are equal to one another, and each is greater than any one of the other terms; and if p is not an integer, the $(q+1)^{\text{th}}$ term is the greatest.

2°. Suppose n is a fraction. Then the series is infinite.

When $r>n+1$, $\dfrac{n+1-r}{r}$ is negative, and therefore the multiplier is $\left(1-\dfrac{n+1}{r}\right)\dfrac{x}{a}$.

If $x>a$, when $r>x\dfrac{n+1}{x-a}$ the multiplier is >1 and the terms go on increasing continually, and therefore there is no greatest term.

If $x<a$, or $=a$, when $r>n+1$ the multiplier <1 and the corresponding terms continually decrease, therefore the greatest term is amongst those for which $r=$ or $<n+1$.

Hence, as in 1°, if p is an integer, the p^{th} and the $(p+1)^{\text{th}}$ etc.

II. Let n be negative. Then the series is infinite.

Put $n=-m$, so that m is positive. Then $\dfrac{n+1}{r}-1$ $=-\left(1+\dfrac{m-1}{r}\right)$. And this being always negative, the multiplier is $\left(1+\dfrac{m-1}{r}\right)\dfrac{x}{a}$.

Let p' be such that $\left(1+\dfrac{m-1}{p'}\right)\dfrac{x}{a}=1$, or $p'=\dfrac{(m-1)x}{a-x}$, and, if p' when positive is not integral, let q' be its integral part.

1°. Suppose $m>1$.

Then $\dfrac{m-1}{r}$ is positive, and $1+\dfrac{m-1}{r}$ never <1, but decreases as r increases.

Hence, if $x>a$, or $=a$, the multiplier never <1, and the terms go on continually increasing.

If $x<a$, the multiplier is >1 when $r<p'$ and <1 when $r>p'$.

Hence, similarly to I. 1°, if p' be an integer, the p'^{th} and the $(p'+1)^{\text{th}}$ etc.

2°. Suppose $m<1$.

Then $\frac{m-1}{r}$ is negative, and $1+\frac{m-1}{r}$ never >1, but increases as r increases.

Hence if $x<a$, or $=a$, the multiplier is never >1, and the terms go on continually decreasing; $\therefore$ the first is the greatest term.

If $x>a$ when $r>p'$, the multiplier is >1, and the terms go on continually increasing, and therefore there is no greatest term.

EXAMPLES.—XII.

Determine, when possible, which is the greatest term in the expansions of the following :—

1. $(1+2x)^{\frac{1}{2}}$.
2. $(3+x)^{\frac{2}{3}}$.
3. $(1+3x)^{-\frac{7}{2}}$.
4. $(1-4x)^{\frac{9}{2}}$.
5. $(2+3x)^{-2}$.
6. $(2+3x)^{-\frac{1}{2}}$.

98. Products are said to be homogeneous when the number of factors in each is the same.

Thus a^3, a^2b, abc are homogeneous products of *three* dimensions.

99. PROP. *To find how many homogeneous products of* r *dimensions can be formed out of the* n *symbols,* a, b, c, . . . *etc.*

[*Mem.* If m be the number of terms in the sum $A+B+C+$ etc., and if we make A, B, etc., each equal to 1, then the sum becomes equal to m.]

We have $\frac{1}{1-ax}=1+ax+a^2x^2+a^3x^3+\text{etc.}$

$$\frac{1}{1-bx}=1+bx+b^2x^2+b^3x^3+\text{etc.}$$

$$\frac{1}{1-cx}=1+cx+c^2x^2+\text{etc.}$$

$$\text{etc.} \quad = \quad \text{etc.};$$

$\therefore$, multiplying, we have

$$\frac{1}{1-ax}\,\frac{1}{1-bx}\ldots=1+(a+b+c+\text{etc.})x$$
$$+(a^2+b^2+\text{etc.}+ab+ac+bc+\text{etc.})x^2+\text{etc.} \qquad (1),$$

the coefficient of x^r being the sum of all the homogeneous products of r dimensions which can be formed out of the symbols.

If now we put $a=b=c=\text{etc.}=1$, each of the products will become equal to 1, and therefore the coefficient of x^r will become the number required. But now the left-hand side of (1) has become $(1-x)^{-n}$, in the expansion of which the coefficient of x^r is $\frac{n(n+1)\ldots(n+r-1)}{\underline{|r}}$. Therefore this is the number required.

100. If p_r denote the coefficient of x^r in the expansion of $(1+x)^{2n}$, then

$$2p_0p_{2n}-2p_1p_{2n-1}+\text{etc.}+2(-1)^{n-1}p_{n-1}p_{n+1}+(-1)^np_n^2$$
$$=(-1)^np_n.$$

We have

$$(1+x)^{2n}=p_{2n}x^{2n}+p_{2n-1}x^{2n-1}+\ldots+p_{n+1}x^{n+1}+\quad p_nx^n$$
$$+p_{n-1}x^{n-1}+\ldots+p_1x+p_0,$$
$$(1-x)^{2n}=p_0\ -\ p_1x+\ldots+(-1)^{n-1}p_{n-1}x^{n-1}+(-1)^np_nx^n$$
$$+(-1)^{n+1}p_{n+1}x^{n+1}+\ldots+p_{2n}x^{2n};$$

$\therefore$, multiplying, we see that

$$p_0p_{2n}-p_1p_{2n-1}+\ldots+(-1)^{n-1}p_{n+1}p_{n-1}+(-1)^np_n^2$$
$$+(-1)^{n+1}p_{n-1}p_{n+1}+\ldots+p_0p_{2n},$$

or $2\{p_0p_{2n}-p_1p_{2n-1}+\ldots+(-1)^{n-1}p_{n+1}p_{n-1}\}+(-1)^np_n^2$

$=$coefficient of x^{2n} in the expansion of $(1-x^2)^{2n}$

$=$coefficient of x^n in the expansion of $(1-x)^{2n}$

$=(-1)^np_n$.

Ex. 2. Prove that the sum of the products of every two consecutive coefficients of an expanded binomial

$$=\frac{1.3.5\ldots(2n-1)}{\underline{|n}}\,\frac{n}{n+1}2^n,$$

n being the index and a positive integer.

Let $(1+x)^n=a_0+a_1x+\;\ldots\;+a_{n-1}x^{n-1}+a_nx^n$;

then $\left(1+\dfrac{1}{x}\right)^n=a_0+\dfrac{a_1}{x}+\;\ldots\;+a_{n-1}\dfrac{1}{x^{n-1}}+\dfrac{a_n}{x^n}$;

$$\therefore\; a_0a_1+a_1a_2+\;\ldots\;+a_{n-1}a_n$$

$=$coefficient of x in expansion of $\dfrac{(1+x)^{2n}}{x^n}$

$=$coefficient of x^{n+1} in expansion of $(1+x)^{2n}$

$$=\frac{\underline{|2n}}{\underline{|n+1}\;\underline{|n-1}}=\frac{\underline{|2n}}{\underline{|n}\;\underline{|n}}\,\frac{n}{n+1}$$

$$=\frac{1.3.5\ldots(2n-1)}{\underline{|n}}2^n\frac{n}{n+1}.$$

EXAMPLES.—XIII.

1. If n be a positive integer, prove that

$$1-2n+\frac{2n(2n-1)}{\underline{|2}}-\ldots+(-1)^{n-1}\frac{2n(2n-1)\ldots(n+2)}{\underline{|n-1}}$$

$$=(-1)^{n-1}\frac{\underline{|2n}}{2(\underline{|n})^2};$$

and if n be an odd number, that

$$1+\frac{2n(2n-1)}{\underline{|2}}+\frac{2n(2n-1)(2n-2)(2n-3)}{\underline{|4}}+\text{etc.}$$

$$+\frac{\underline{|2n}}{\underline{|n-1}\;\underline{|n+1}}=2^{2n-2}.$$

2. If n be an even integer, show that

$$\frac{1}{\lfloor 1 \lfloor n-1}+\frac{1}{\lfloor 3 \lfloor n-3}+\ .\ .\ .\ +\frac{1}{\lfloor n-1 \lfloor 1}=\frac{2^{n-1}}{\lfloor n}.$$

3. If $f(r)$ be the coefficient of x^r in the expansion of $(1+x)^n$, prove that

$$f(r)+\frac{\lfloor m}{\lfloor m-1}f(r-1)+\frac{\lfloor m}{\lfloor m-2 \lfloor 2}f(r-2)+\ldots+f(r-m)=\frac{\lfloor m+n}{\lfloor r \lfloor m+n-r}.$$

4. If $a_0, a_1, a_2 \ldots$ be the coefficients of $(1+x)^n$, show that

(1) $a_0a_r+a_1a_{r+1}+a_2a_{r+2}+\ldots+a_{n-r}a_n=\dfrac{\lfloor 2n}{\lfloor n-r \lfloor n+r}.$

(2) $a_0a_{2r}-a_1a_{2r-1}+a_2a_{2r-2}-\ .\ .\ .\ +(-1)^{r-1}a_{r-1}a_{r+1}$

$$+(-1)^r\tfrac{1}{2}a_r^2=\tfrac{1}{2}(-1)^r\frac{\lfloor n}{\lfloor n-r \lfloor r}.$$

(3) $a_0a_r-a_1a_{r+1}+a_2a_{r+2}-\ .\ .\ .\ +(-1)^{n-r}a_{n-r}a_n$

$$=(-1)^{\frac{n-r}{2}}\frac{\lfloor n}{\lfloor \frac{1}{2}(n-r) \lfloor \frac{1}{2}(n+r)}.$$

Examine the condition as to the values of n and r in this last case, and find the value of the expression when this condition is not satisfied.

5. If $(x+a)^n=p_0x^n+p_1x^{n-1}a+p_2x^{n-2}a^2+\ .\ .\ .,$

$(x+a)^m=q_0x^m+q_1x^{m-1}a+q_2x^{m-2}a^2+\ .\ .\ .,$

m and n being positive integers, find the value of

$$p_0q_r+p_1q_{r-1}+p_2q_{r-2}+\ .\ .\ .$$

6. If $(n)_r$ be the coefficient of x^r in the expansion of $(1+x)^n$, prove that $(2n)_r=2(n)_r+2(n)_1(n)_{r-1}+\ .\ .\ .$, n being a positive integer.

Write down the last term for the two cases in which r is odd and even.

7. Find the sum of the squares of the coefficients of the expansion of $(1+x)^n$, n being a positive integer.

101. The above are examples of the equality of coefficients (Art. 87) in two *finite* series, which are known to be always equal to one another from their being expansions of the same function.

We shall now give some examples of this equality in two infinite series, which are both *convergent*, and thus are known to be equal for the same reason (Art. 94).

102. *Ex.* 1. Prove that

$$1-n^2+n^2\cdot\frac{n^2-1}{2^2}+n^2\frac{n^2-1}{2^2}\cdot\frac{n^2-2^2}{3^2}+\text{ etc. }=0,$$

n being a positive integer.

We have

$$(1+x)^n=1+nx+\frac{n(n-1)}{2}x^2+\frac{n(n-1)(n-2)}{1.2.3}x^3+\text{ etc.}\quad (1),$$

$$\left(1+\frac{1}{x}\right)^{-n}=1-\frac{n}{x}+\frac{n(n+1)}{2}\frac{1}{x^2}-\frac{n(n+1)(n+2)}{1.2.3}\frac{1}{x^3}+\text{etc.}\quad (2).$$

Multiplying we see that $1-n^2+\frac{n^2(n^2-1)}{2^2}-$ etc. is the coefficient of x^0 in the expansion of $(1+x)^n.\left(1+\frac{1}{x}\right)^{-n}$;

$$i.e.,\text{ of }(1+x)^n.\frac{x^n}{(1+x)^n},\text{ or of }x^n;$$

$$\therefore\ 1-n^2+\frac{n^2(n^2-1)}{2^2}-\text{ etc. }=0.$$

Here we were able to equate co-efficients, for when x is sufficiently large ($i.e. > 1$), the series in (2) is convergent, and therefore the series formed by multiplying it by the finite series on the right hand of (1), is convergent.

Ex. 2. If $p_r=\frac{1.3.5\ \ldots\ (2r-1)}{2.4.6\ \ldots\ 2r}$, show that

$$p_{2n+1}+p_1p_{2n}+\ \ldots\ +p_np_{n+1}=\frac{1}{2}.$$

Here p_r is the coefficient of x^r in the expansion of $(1-x)^{-\frac{1}{2}}$;

$$\therefore\ (1-x)^{-\frac{1}{2}}=1+p_1x+p_2x^2+\ \ldots\ +p_{2n+1}x^{2n+1}+\text{ etc.}\quad (1),$$

$$(1-x)^{-\frac{1}{2}}=1+p_1x\ \ldots\ldots\ldots\ +p_{2n+1}x^{2n+1}+\text{ etc.}\quad (2).$$

Multiplying we see that

$$p_{2n+1}+p_1p_{2n}+\ \cdot\ \cdot\ \cdot\ +p_np_{n+1}+p_{n+1}p_n+\ \cdot\ \cdot\ \cdot\ +p_{2n}p_1+p_{2n+1}$$

$=$co-efficient of x^{2n+1} in the expansion of $(1-x)^{-1}$

$$=\quad 1;$$

$$\therefore\ p_{2n+1}+p_1p_{2n}+\ \cdot\ \cdot\ \cdot\ +p_np_{n+1}=\tfrac{1}{2}.$$

It will easily be seen that the series in (1) and (2) is convergent, and therefore also its square, when x is <1.

EXAMPLES.—XIV.

1. Prove that

$$\frac{1.3.5\ldots(2r-1)}{|\underline{r}}+\frac{1.3.5\ldots(2r-3)}{|\underline{r-1}}\,\frac{3}{1}+\frac{1.3.5\ldots(2r-5)}{|\underline{r-2}}\,\frac{3.5}{|\underline{2}}$$

$$+\text{ etc.}\qquad =2^r(1+r).$$

2. The sum of the first $\overline{r+1}$ coefficients of the expansion of $(1-x)^{-m}$ is $\dfrac{|\underline{m+r}}{|\underline{m}\,|\underline{r}}$, m being a positive integer.

3. Prove that

$$1+3n+\frac{3.4}{1.2}\,\frac{n(n-1)}{|\underline{2}}+\frac{4.5}{1.2.}\,\frac{n(n-1)(n-2)}{|\underline{3}}+\text{etc.}$$

$$=2^{n-3}(n^2+7n+8).$$

Write down the last term of the series on the left side.

4. If $p_r=\dfrac{-1.2.5.8\ \ldots\ (3r-4)}{3.6.9.12\ \ldots\ 3r}$, show that

$$p_{2n+1}+p_{2n}p_1+p_{2n-1}p_2+p_{2n-2}p_3+\ldots p_{n+1}p_n=\frac{-2.1.4\ldots(6n-5)}{3.6.9\ldots6n}.$$

5. If $\phi(r,n)=\dfrac{|\underline{n+r-1}}{|\underline{n-1}\,|\underline{r}}$, prove that

(1) $\phi(r,n+1)=1+\phi(1,n)+\phi(2,n)+\ \cdot\ \ \cdot\ +\phi(r,n)$;

(2) $\phi(r+1,n)=\phi(r,1)+\phi(r,2)+\ \cdot\ \ \cdot\ +\phi(r,n)$.

103. Since $\sqrt{5}$ lies between two integers, viz., 2 and 3, $3+\sqrt{5}=5+$ some fraction; $\therefore (3+\sqrt{5})^n=$ an integer $+$ some fraction, n being a positive integer. Required to find expressions for the integral and fractional parts of $(3+\sqrt{5})^n$.

Denote them by a and f respectively, so that

$$(3+\sqrt{5})^n=a+f.$$

Now $3-\sqrt{5}$, and $\therefore (3-\sqrt{5})^n$, is a proper fraction.

Denote it by f', so that

$$(3-\sqrt{5})^n=f'.$$

Now $(3+\sqrt{5})^n=3^n+\frac{n(n-1)}{1.2}3^{n-2}5+\text{ etc.}$

$$+\sqrt{5}\left\{n3^{n-1}+\frac{n(n-1)(n-2)}{1.2.3}3^{n-3}5+\text{ etc.}\right\},$$

$$(3-\sqrt{5})^n=3^n+\frac{n(n-1)}{1.2}3^{n-2}5+\text{etc.}-\sqrt{5}\left\{n3^{n-1}+\text{ etc.}\right\};$$

$$\therefore a+f+f'=2\left\{3^n+\frac{n(n-1)}{1.2}3^{n-2}5+\text{ etc.}\right\}$$

$=$ an integer, since the coefficients $\frac{n(n-1)}{1.2}$, etc., being the number of combinations of n things taken 2, 4, etc. times together, must be integral.

Hence a being an integer, $f+f'$ must be one also, or zero.

Now f and f' being each positive and less than 1, $f+f'$ must be greater than 0 and less than 2;

$$\therefore \text{ it can only be equal to } 1;$$

$$\therefore a=2\left\{3^n+\frac{n(n-1)}{1.2}3^{n-2}5+\text{ etc.}\right\}-1.$$

Also $f=1-f'=1-(3-\sqrt{5})^n$

Obs. $3^n+\frac{n(n-1)}{1.2}3^{n-2}5+$ etc. being an integer,

$2\left\{3^n+\frac{n(n-1)}{1.2}3^{n-2}5+\text{ etc.}\right\}$, or $a+1$, is an even integer;

$\therefore a$ is an odd integer.

COR. The rational part of $(3+\sqrt{5})^n=3^n+\frac{n(n-1)}{1.2}3^{n-2}5+$ etc.

$$=\tfrac{1}{2}(a+1),$$

and the irrational part $=\sqrt{5}\{n3^{n-1}+\text{etc.}\}$

$$=\frac{1}{2}\{(3+\sqrt{5})^n-(3-\sqrt{5})^n\}$$

$$=\frac{1}{2}\{a+f-f'\}$$

$$=\frac{1}{2}\{a-1+2f\}.$$

EXAMPLES.—XV.

Find the integral parts of the following.

1. $(2+\sqrt{3})^n$.
2. $(3+\sqrt{7})^n$.
3. $(3+\sqrt{6})^n$.
4. $(7+3\sqrt{5})^n$.
5. $(3+2\sqrt{2})^n$.
6. $(5+3\sqrt{2})^n$.

104. To find the integral part of $(\sqrt{5}+2)^n$.

Let a and f denote the integral and fractional parts respectively, so that $(\sqrt{5}+2)^n=a+f$.

Also $(\sqrt{5}-2)^n=f'$, a proper fraction.

1°. Let n be odd.

Then $a+f-f'=(\sqrt{5}+2)^n-(\sqrt{5}-2)^n$

$$=2\left\{n5^{\frac{n-1}{2}}2+\frac{n(n-1)(n-2)}{1.2.3}5^{\frac{n-3}{2}}2^3+\text{ etc. }\right\}=\text{an integer}.$$

But a is an integer; $\therefore f-f'$ is an integer, or zero.

Now f and f' being each >0 and <1, $f-f'$ cannot be numerically so great as 1; $\therefore f-f'=0$; $\therefore f=f'$;

$$\text{and } a=2\{n.5^{\frac{n-1}{2}}2+\text{ etc.}\}$$

2°. Let n be even.

Then $\alpha+f+f'=2\left\{5^{\frac{n}{2}}+\frac{n(n-1)}{1.2}5^{\frac{n-2}{2}}2^2+\text{ etc. }\right\}$;

and, as in Art. 103, $f+f'=1$;

$$\therefore \alpha=2\left\{5^{\frac{n}{2}}+\frac{n(n-1)}{1.2}5^{\frac{n-2}{2}}2^2+\text{ etc. }\right\}-1.$$

EXAMPLES.—XVI.

Find the integral parts of the following:—(1) when n is even, and (2) when it is odd.

1. $(\sqrt{3}+1)^n$. 2. $(\sqrt{6}+2)^n$. 3. $(\sqrt{10}+3)^n$. 4. $(2\sqrt{3}+3)^n$.
5. $(3\sqrt{2}+4)^n$. 6. $(3\sqrt{5}+6)^n$. 7. $(4\sqrt{2}+5)^n$.

Find the integral part of the following when n is even.

8. $(\sqrt{3}+\sqrt{2})^n$. 9. $(\sqrt{7}+\sqrt{5})^n$. 10. $(2\sqrt{3}+\sqrt{11})^n$.

11. If $(\sqrt{2}+1)^{2m+1}=a+f$, where m and a are positive integers, and f a positive proper fraction, then $f(a+f)=1$.

12. Prove the same thing, if $(5\sqrt{2}+7)^{2m+1}=a+f$.

13. Prove that $f(a+f)=2^{2m+1}$, if $(3\sqrt{3}+5)^{2m+1}=a+f$.

105. Expand $\frac{x+x^2}{(1-x)^3}$ in a series of ascending powers of x.

We have $\frac{x+x^2}{(1-x)^3}=(x+x^2)(1-x)^{-3}$

$$=(x+x^2)\left(1+3x+\frac{3.4}{1.2}x^2+\frac{4.5}{1.2}x^3+\text{ etc.}\right.$$

$$\left.+\frac{(r+1)(r+2)}{1.2}x^r+\text{ etc.}\right)$$

$$=x+4x^2+9x^3+\text{ etc.},$$

the coefficient of x^r being $\frac{(r-1)r}{1.2}+\frac{r(r+1)}{1.2}=r^2$.

106. If a_r be the coefficient of x^r in the expansion of $(1+x+x^2)^n$, then

$$a_0a_{2r}-a_1a_{2r+1}+a_2a_{2r-2}- \dots +(-1)^{r-1}a_{r-1}a_{r+1} = \frac{a_r}{2}\left\{1-(-1)^r a_r\right\}.$$

We have $(1+x+x^2)^n=a_0+a_1x+a_2x^2+ \dots +a_{2r}x^{2r}+$ etc.,
and $(1-x+x^2)^n=a_0-a_1x+a_2x^2- \dots +a_{2r}x^{2r}-$etc.;

$$\therefore a_0a_{2r}-a_1a_{2r-1}+ \text{etc.} +(-1)^{r-1}a_{r-1}a_{r+1}+(-1)^r a_r^2 +(-1)^{r+1}a_{r+1}a_{r-1}+ \text{etc.}+a_{2r}a_0$$

$=$coeff. of x^{2r} in the expansion of the product
$(1+x+x^2)^n(1-x+x^2)^n$,
i.e., of $\{(1+x^2)^2-x^2\}^n$, or of $(1+x^2+x^4)^n$,
$=$ coeff. of x^r in the expansion of $(1+x+x^2)^n=a_r$;

$$\therefore a_0a_{2r}-a_1a_{2r-1}+ \dots +(-1)^{r-1}a_{r-1}a_{r+1}=\frac{a_r}{2}-\frac{(-1)^r}{2}a_r^2.$$

EXAMPLES.—XVII.

1. If ${}_nH_r$ be the number of homogeneous products of n symbols taken r together, prove that

(1) $1+{}_nH_1+{}_nH_2+ \dots +{}_nH_r={}_{n+1}H_r$;

and (2) ${}_nH_r-{}_nH_{r-1}C_1+{}_nH_{r-2}C_2- \dots =0$,

where C_r denotes the number of combinations of n things taken r together.

2. If ${}_nC_r$ and ${}_nH_r$ represent the expressions

$$\frac{n(n-1)\dots(n-r+1)}{\underline{|r}}, \quad \frac{n(n+1)\dots(n+r-1)}{\underline{|r}}$$

respectively; show that

$${}_nC_{4r}+{}_nC_{4r-2}\cdot{}_nC_1+{}_nC_{4r-4}\cdot{}_nC_2+\dots+{}_nC_2\cdot{}_nC_{2r-1}+{}_nC_{2r}$$
$$={}_nH_{4r}-{}_nH_{4r-4}\cdot{}_nC_1+{}_nH_{4r-8}\cdot{}_nC_2-\dots+(-1)^{r-1}{}_nH_4\cdot{}_nC_{r-1}+(-1)^r{}_nC_r$$

3. Show that the sum of the products r together of the n quantities $a, a^2, \ldots a^n$ is

$$\frac{(a^{r+1}-1)(a^{r+2}-1)\ldots(a^n-1)}{(a-1)(a^2-1)\ldots(a^{n-r}-1)}a^{\frac{r(r+1)}{2}}.$$

4. In the expansion of $(a_1+a_2+\ldots+a_p)^n$, if n is a whole number, and $p>n$, prove that the coefficient of any term, in which none of the quantities $a_1, a_2, \ldots$ appears more than once, is equal to $\underline{|n}$.

Prove that

$$(n^2-2)^{\frac{1}{2}}=\frac{n^2-1}{n}\left\{1-\frac{1}{2}\frac{1}{(n^2-1)^2}-\frac{1}{8}\frac{1}{(n^2-1)^4}-\frac{1}{16}\frac{1}{(n^2-1)^6}\right.$$
$$\left.-\frac{5}{128}\frac{1}{(n^2-1)^8}\cdots\right\}.$$

5. Show that the remainder after n terms of the expansion of $(1-x)^{-2}$ is

$$\frac{(n+1)x^n-nx^{n+1}}{(1-x)^2}$$

6. Find the first 7 terms of the expansion of $(1+x+x^2-x^3)^{10}$.

7. Prove that

$$1+\frac{1}{6}\frac{n^2}{1}+\left(\frac{1}{6}\right)^2\left\{\frac{n(n-1)}{\underline{|2}}\right\}^2+\left(\frac{1}{6}\right)^3\left\{\frac{n(n-1)(n-2)}{\underline{|3}}\right\}^2+\ldots$$
$$=\left(\frac{7}{6}\right)^n\left\{1+\frac{n(n-1)}{1}\left(\frac{6}{7^2}\right)+\frac{n(n-1)(n-2)(n-3)}{(\underline{|2})^2}\left(\frac{6}{7^2}\right)^2+\ldots\right\}.$$

8. If a_r be the coefficient of x^r in the expansion of $(1+x)^n$, prove that, if k be less than n,

$$\frac{\underline{|n}}{\underline{|n-k}}-a_1\frac{\underline{|n-1}}{\underline{|n-k-1}}+a_2\frac{\underline{|n-2}}{\underline{|n-k-2}}-\text{ etc.}=0;$$

there being $n-k+1$ terms in the series.

9. Sum to infinity the series

$$1+\frac{1+x}{1.2}+\frac{(1+x)(1+2x)}{1.2.3}+\frac{(1+x)(1+2x)(1+3x)}{1.2.3.4}+\ldots$$

10. Prove that the sum of the numerical coefficients of all

terms containing a_r^p in the expansion of $(a_1+a_2+ \ldots +a_n)^m$ is

$$\frac{m(m-1) \ldots (m-p+1)}{\underline{|p}}(n-1)^{m-p}.$$

11. The coefficient of d^{n-3} in the expansion of $(a+b+c+d)^n$ is

$$\frac{n(n-1)(n-2)}{6}(a^3+b^3+c^3+3b^2c+3bc^2+ \text{ etc. } \ldots +6abc),$$

and the number of terms in the coefficient of d^{n-5} is 21.

12. Determine the coefficient of any power of x in the expansion of

$$\frac{(n-m+1)x(1-x)-x^{m+1}+x^{n+2}}{(1-x)^2}$$

according to ascending powers of x, where m and n are positive integers and $n+2>m+1$.

13. If S_0, S_e be respectively the sums of the coefficients of the odd and even powers of x in the expansion of $(1+x+ \ \ldots \ +x^n)^m$, prove that $S_0=S_e$ when n is odd, and $S_0=S_e-1$ when n is even.

14. Show that if t_r denote the middle term of the expansion of $(1+x)^{2r}$, then $t_0+t_1+t_2+ \ \ldots \ =(1-4x)^{-\frac{1}{2}}$.

15. Find the coefficient of x^r in the expansion of

$$(1+2x+3x^2+4x^3+ \text{etc})^2.$$

IX

Multinomial Theorem.

107. In [Art. 418] it was shown how, by successive applications of the Binomial Theorem, to express any integral power of a trinomial by means of a series. We proceed to the general extension of the same method.

Prop. *To find the general term of the series which is the expansion of* $(a+b+c+\text{ etc. }+t+u)^n$, n *being unrestricted.*

From [Art. 419] we see that the general term of the expansion of $(x+y)^m$ is $\dfrac{m(m-1)\ldots(m-r+1)}{\underline{|r}}x^{m-r}y^r$, r being 0, or any positive integer (not greater than m, if m is a positive integer).

The method consists in repeated applications of this theorem.

Now $(a+b+c+\text{ etc.})^n=a^n+na^{n-1}(b+c+\text{ etc.})+\text{ etc.}$,
the general term of the series being

$$\frac{n(n-1)\ \ .\ \ .\ \ .\ \ (n-\alpha+1)}{\underline{|\alpha}}a^{n-\alpha}(b+c+\text{etc.})^\alpha,$$

where α is 0, or any positive integer, except when n is a positive integer, and then α must not be greater than n.

Let M denote $n(n-1)\ .\ .\ .\ (n-\alpha+1)$ for shortness.

Again $\dfrac{M}{\underline{|\alpha}}a^{n-\alpha}(b+c+\text{etc.})^\alpha=\dfrac{M}{\underline{|\alpha}}a^{n-\alpha}\{b^\alpha+\alpha b^{\alpha-1}(c+\text{etc.})+\text{etc.}\}$,

the general term being $\dfrac{M}{\underline{|\alpha}}a^{n-\alpha}\dfrac{\underline{|\alpha}}{\underline{|\alpha-\beta}\,\underline{|\beta}}b^{\alpha-\beta}(c+\text{etc.})^\beta$

$$=\frac{M}{\underline{|\alpha-\beta}\,\underline{|\beta}}a^{n-\alpha}b^{\alpha-\beta}(c+\text{ etc.})^\beta,$$

where β is 0 or any positive integer not greater than α.

Again $\dfrac{M}{\underline{|\alpha-\beta}\,\underline{|\beta}}a^{n-\alpha}b^{\alpha-\beta}(c+\text{etc.})^\beta$ expands into a series of which the general term is $\dfrac{M}{\underline{|\alpha-\beta}\,\underline{|\beta-\gamma}\,\underline{|\gamma}}a^{n-\alpha}b^{\alpha-\beta}c^{\beta-\gamma}(d+\text{etc.})^\gamma$.

And this process can be carried on until we find for the final general term $\dfrac{M}{|\underline{\alpha-\beta} \ldots |\underline{\sigma-\tau}\, |\underline{\tau}} a^{n-\alpha}b^{\alpha-\beta} \ldots t^{\sigma-\tau}u^{\tau}$.

Where $\alpha, \beta \ldots \sigma, \tau$ are zero, or such positive integers that each is less than, or equal to, the preceding.

Note.—The sum of all the indices is equal to n.

We may simplify this formula by putting

$$\left.\begin{array}{l} \alpha-\beta=\beta', \\ \beta-\gamma=\gamma', \\ \text{etc.}=\text{etc.} \\ \sigma-\tau=\tau'; \end{array}\right\} \begin{array}{l} \therefore n-\alpha+\beta'+\gamma'+\ldots\tau'+\tau=n; \\ \therefore \beta'+\gamma'+\ldots\tau'+\tau=\alpha. \end{array}$$

Then the general term is $\dfrac{M}{|\underline{\beta'}\, |\underline{\gamma'} \ldots |\underline{\tau'}\, |\underline{\tau}} a^{n-\alpha}b^{\beta'} \ldots t^{\tau'}u^{\tau}$

where $\beta', \ldots \tau'$ are all positive integers.

108. If n be a positive integer, put $n-\alpha=\alpha'$, a positive integer, and $M=\dfrac{|\underline{n}}{|\underline{\alpha'}}$, and the general term becomes

$$\frac{|\underline{n}}{|\underline{\alpha'},\, |\underline{\beta'} \ldots |\underline{\tau'}\, |\underline{\tau}} a^{\alpha'}b^{\beta'} \ldots t^{\tau'}u^{\tau},$$

and $n=\alpha'+\beta'+\gamma'+\ldots+\tau'+\tau$.

109. *Ex.* 1. Find the terms of the expansion of $(a+bx-cx^2)^6$ which involve x^7.

Here the general term is

$$\frac{|\underline{6}}{|\underline{\alpha'}\, |\underline{\beta'}\, |\underline{\gamma'}} a^{\alpha'}(bx)^{\beta'}(-cx^2)^{\gamma'}$$

$$=\frac{|\underline{6}}{|\underline{\alpha'}\, |\underline{\beta'}\, |\underline{\gamma'}} a^{\alpha'}b^{\beta'}(-c)^{\gamma'}x^{\beta'+2\gamma'}$$

where $\alpha'+\beta'+\gamma'=6 \ldots$ (1), and we want all the terms in which $\beta'+2\gamma'=7 \ldots$ (2),

Hence we have to solve (1) and (2) in positive integers.

The solutions will be found to be

$$\begin{array}{llll} \beta'=1, & 3, & 5 & \\ \gamma'=3, & 2, & 1 & \text{from (2)} \\ \alpha'=2, & 1, & 0 & \text{,, (1).} \end{array}$$

Hence the required terms are

$$\frac{-\underline{|6}}{\underline{|2}\,\underline{|3}}a^2bc^3x^7,\quad \frac{\underline{|6}}{\underline{|2}\,\underline{|3}}ab^3c^2x^7,\quad \frac{-\underline{|6}}{\underline{|5}}b^5cx^7.$$

In writing down expansions by the Multinomial Theorem, we arrange them according to ascending powers of some common letter.

Often the sum of all the terms involving the same power of this letter is called one term, viz., the term involving that power.

Thus, above, $\underline{|6}\,bcx^7\left(-\frac{1}{\underline{|2}\,\underline{|3}}a^2c^2+\frac{1}{\underline{|2}\,\underline{|3}}ab^2c-\frac{1}{\underline{|5}}b^4\right)$

is the term involving x^7.

Ex. 2. Find the term involving x^5 in the expansion of $(2-2x-x^3+3x^4)^{-\frac{1}{6}}$.

The general term is

$$\frac{-\frac{1}{6}(-\frac{1}{6}-1)\ldots(-\frac{1}{6}-\alpha+1)}{\underline{|\beta'}\,\underline{|\gamma'}\,\underline{|\delta'}}2^{(-\frac{1}{6}-\alpha)}(-2x)^{\beta'}(-x^3)^{\gamma'}(3x^4)^{\delta'},$$

$$=\frac{(-1)^{\alpha+\beta+\gamma}}{2^{\frac{1}{6}}}\,\frac{7.13\ .\ .\ .\ (6\alpha-5)}{6^{\alpha}\,\underline{|\beta'}\,\underline{|\gamma'}\,\underline{|\delta'}}2^{-\alpha+\beta'}3^{\delta'}x^{\beta'+3\gamma'+4\delta'}.$$

Here $\beta'+\gamma'+\delta'=\alpha, \quad . \quad . \quad . \quad (1),$

$\beta'+3\gamma'+4\delta'=5, \quad . \quad . \quad . \quad (2).$

The solutions of these equations are

$$\begin{array}{llll} \delta'=0, & \beta'=2, & 5, & \\ & \gamma'=1, & 0, & \text{from (2)}, \\ & \alpha=3, & 5, & \text{,, (1).} \\ \delta'=1, & \beta'=1, & & \\ & \gamma'=0, & \text{from (2)}, & \\ & \alpha=2, & \text{,, (1).} & \end{array}$$

Therefore the required term is

$$\frac{x^5}{2}\left\{\frac{7.13}{6^3\,\underline{|2}}2^{-1}+\frac{7.13.19.25}{6^5\,\underline{|5}}-\frac{7}{6^2}2^{-1}3\right\}.$$

EXAMPLES.—XVIII.

1. Find the coefficient of x^4 in the expansion of $(1+2x-3x^2+x^3)^8$.

2. Find the coefficient of x^5 in $(2-x+x^2-3x^3)^4$.

3. Find the coefficient of x^5 in $\{x-\frac{1}{3}x^3+\frac{1}{5}x^5-\frac{1}{7}x^7+\text{etc.}\}^3$.

4. Prove that the sum of the coefficients, taken positively, of the expansion of $(1-x+x^2)^n$ is 3^n, n being a positive integer; and that the sum of the squares of the same coefficients is

$$\frac{\lfloor 2n}{(\lfloor n)^2}\left\{1+\frac{n^2}{\lfloor 2}+\frac{n^2(n-1)^2}{\lfloor 4}+\ldots\right\}.$$

5. Find the coefficients of x^{2n}, x^{2n-1}, x^{2n-2} in the expansion of $(1-x+x^2)^{2n}$.

6. Find the coefficient of x^3 in the expansion of $(1-2x+3x^2)^4$.

7. " " x^4 " " $(1+x-2x^2)^{-3}$.

8. " " x^5 " " $(1+3x-2x^2)^{\frac{3}{2}}$.

9. " " x^4 " " $(1-x-3x^3+2x^4)^{\frac{3}{2}}$.

10. " " x^3 " " $(a+bx+cx^2)^{\frac{1}{2}}$.

11. " " abc^2 " " $(a+bx+cx^2)^4$.

12. " " $a^{\frac{1}{2}}b^2$ " " $(a+2bx+cx^2)^{\frac{5}{2}}$.

13. Find the number of terms in the expansion of $(a+b+c)^6$.

14. " " $(a+b+c+d)^4$.

15. Expand $(1-2x-2x^2)^4$ to 5 terms.

16. " $(1-ax+x^2)^{\frac{5}{2}}$ " 4 terms.

X

Inequalities.

110. Prop. *If* $a_1, a_2, \ldots a_n$ *be* n *positive numbers, not all equal, then* $\dfrac{a_1+a_2+\;\ldots\;+a_n}{n} > (a_1 a_2\;\ldots\;a_n)^{\frac{1}{n}}$.

For convenience we will assume that the n numbers are in ascending order of magnitude when written down as above.

Let x be a positive number; we have

$$\left(1+\frac{x}{n}\right)^n = 1+x+\frac{1-\frac{1}{n}}{2}x^2+\frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{\underline{|3}}x^3+\text{ etc.},$$

$$\left(1+\frac{x}{n-1}\right)^{n-1} = 1+x+\frac{1-\frac{1}{n-1}}{2}x^2+\frac{\left(1-\frac{1}{n-1}\right)\left(1-\frac{2}{n-1}\right)}{\underline{|3}}x^3+\text{ etc.}$$

Now $1-\frac{1}{n} > 1-\frac{1}{n-1}$, $1-\frac{2}{n} > 1-\frac{2}{n-1}$, etc.; hence every term, after the second, in the first series, is greater than the corresponding term in the second; also the first series contains one more term than the second, and, since x is positive, all the terms are positive;

$$\therefore \left(1+\frac{x}{n}\right)^n > \left(1+\frac{x}{n-1}\right)^{n-1}, \quad . \quad . \quad (1).$$

Now

$$\left(\frac{a_1+a_2+\ldots+a_n}{n}\right)^n = a_1{}^n\left\{1+\frac{a_1+a_2+\ldots+a_n-na_1}{n.a_1}\right\}^n$$

$$> a_1{}^n\left\{1+\frac{a_1+a_2+\ldots+a_n-na_1}{(n-1)a_1}\right\}^{n-1},$$

by (1), since $\dfrac{a_1+a_2+\ldots+a_n-na_1}{a_1}$ is a positive number;

$$\therefore \left(\frac{a_1+a_2+\ldots+a_n}{n}\right)^n > a_1\left(\frac{a_2+a_3+\ldots+a_n}{n-1}\right)^{n-1}.$$

$$\text{Similarly} \left(\frac{a_2+a_3+\ldots+a_n}{n-1}\right)^{n-1} > a_2\left(\frac{a_3+a_4+\ldots+a_n}{n-2}\right)^{n-2},$$

$$\text{etc.} > \text{etc.}$$

$$\left(\frac{a_{n-2}+a_{n-1}+a_n}{3}\right)^3 > a_{n-2}\left(\frac{a_{n-1}+a_n}{2}\right)^2,$$

$$\left(\frac{a_{n-1}+a_n}{2}\right)^2 > a_{n-1}a_n;$$

$$\therefore, \text{ multiplying, } \left(\frac{a_1+a_2+\ldots+a_n}{n}\right)^n > a_1a_2 \ldots a_{n-1}a_n;$$

$$\therefore \frac{a_1+a_2+\ldots+a_n}{n} > (a_1a_2 \ldots a_n)^{\frac{1}{n}}. \qquad \text{Q.E.D.}$$

This theorem is sometimes thus stated:—"*The Arithmetic mean of any number of positive numbers is greater than their Geometric mean.*" We have given Mr. Thacker's Proof from the *Cambridge and Dublin Mathematical Journal*, vol. vi., p. 81.

111. Prop. *If* $a_1, a_2, \ldots, a_n$ *be* n *positive numbers, not all equal, then* $\frac{a_1^m+a_2^m+\ldots+a_n^m}{n} \lesseqgtr \left(\frac{a_1+a_2+\ldots+a_n}{n}\right)^m$, *according as* m *does, or does not, lie between* 0 *and* 1, *except when* m=0 or 1, *and then the inequality becomes an equality.*

The truth of the statement with regard to the two exceptional cases is evident.

I. For *two* numbers; denote them by a and b. We have

$$a^m+b^m=\left(\frac{a+b}{2}+\frac{a-b}{2}\right)^m+\left(\frac{a+b}{2}-\frac{a-b}{2}\right)^m$$

$$=2\left\{\left(\frac{a+b}{2}\right)^m+\frac{m(m-1)}{1.2}\left(\frac{a+b}{2}\right)^{m-2}\left(\frac{a-b}{2}\right)^2\right.$$

$$\left.+\frac{m(m-1)(m-2)(m-3)}{\underline{|4}}\left(\frac{a+b}{2}\right)^{m-4}\left(\frac{a-b}{2}\right)^4+\ldots\right\};$$

$$\therefore \frac{a^m+b^m}{2}-\left(\frac{a+b}{2}\right)^m=\frac{m(m-1)}{1.2}\left(\frac{a+b}{2}\right)^{m-2}\left(\frac{a-b}{2}\right)^2$$

$$+\frac{m(m-1)(m-2)(m-3)}{\underline{|4}}\left(\frac{a+b}{2}\right)^{m-4}\left(\frac{a-b}{2}\right)^4+\ldots$$

This series is convergent (*Ex.* V. 11), and therefore represents the arithmetical magnitude of the left-hand side.

The sign of each term is the same as that of the coefficients $m(m-1)$, etc.; hence

(1) when *m is negative*, each of these coefficients is positive;

$$\therefore \frac{a^m+b^m}{2} > \left(\frac{a+b}{2}\right)^m.$$

(2) when *m is positive and* <1, each of the coefficients is negative;

$$\therefore \frac{a^m+b^m}{2} < \left(\frac{a+b}{2}\right)^m.$$

(3) when *m is positive and* >1, some coefficients are positive and some negative, so that the above series does not readily help us to determine the sign of $\frac{a^m+b^m}{2}-\left(\frac{a+b}{2}\right)^m$. This, however, we can do as follows.

Put $m=\frac{p}{q}$, where p and q are positive integers and $p>q$; then $a^{\frac{p}{q}}$, $b^{\frac{p}{q}}$ being two positive numbers, and $\frac{q}{p}<1$, we have by (2)

$$\left(\frac{a^{\frac{p}{q}}+b^{\frac{p}{q}}}{2}\right)^{\frac{q}{p}} > \frac{(a^{\frac{p}{q}})^{\frac{q}{p}}+(b^{\frac{p}{q}})^{\frac{q}{p}}}{2}$$

$$> \frac{a+b}{2};$$

$$\therefore \frac{a^{\frac{p}{q}}+b^{\frac{p}{q}}}{2} > \left(\frac{a+b}{2}\right)^{\frac{p}{q}}, \text{ Art. 39, (3).}$$

Hence the theorem is true for any *two* positive unequal numbers. We will now extend it for *any* number of positive unequal numbers, when *m is* <0 *or* >1.

II. For any numbers, of which the number is a power of 2.

Now $a_1^m+a_2^m>2\left(\frac{a_1+a_2}{2}\right)^m$, $a_3^m+a_4^m>2\left(\frac{a_3+a_4}{2}\right)^m$, by I.;

$$\therefore\ a_1^m+a_2^m+a_3^m+a_4^m>2\left\{\overline{\frac{a_1+a_2}{2}}\Big|^m+\overline{\frac{a_3+a_4}{2}}\Big|^m\right\}$$

$$>2.2\left(\frac{\frac{a_1+a_2}{2}+\frac{a_3+a_4}{2}}{2}\right)^m,\ \text{by I.,}$$

$$>4\left(\frac{a_1+a_2+a_3+a_4}{4}\right)^m;$$

$\therefore$ the theorem is true for 4 numbers, m being <0 or >1.

Similarly we can prove its truth for 8, then for 16, and so on for any numbers, of which the number is a power of 2.

III. We will now show that, if it is true when $n=x$, it is also true when $n=x-1$, m still being <0 or >1.

We have

$$\left(\frac{a_1+a_2+\ldots+a_{x-1}}{x-1}\right)^m=\left(\frac{a_1+a_2+\ldots+a_{x-1}+\frac{a_1+a_2+\ldots+a_{x-1}}{x-1}}{x}\right)^m$$

$$<\frac{a_1^m+a_2^m+\ldots+a^m_{x-1}+\left(\frac{a_1+a_2+\ldots+a_{x-1}}{x-1}\right)^m}{x}$$

$$\therefore\ x\left(\frac{a_1+a_2+\ldots+a_{x-1}}{x-1}\right)^m<a_1^m+a_2^m+\ldots+a^m_{x-1}$$
$$+\left(\frac{a_1+a_2+\ldots+a_{x-1}}{x-1}\right)^m$$

$$\therefore\ \left(\frac{a_1+a_2+\ldots+a_{x-1}}{x-1}\right)^m<\frac{a_1^m+a_2^m+\ldots+a^m_{x-1}}{x-1}.$$

By II. the theorem is true when n is any power of 2, say 2^r, where r is a positive integer; therefore, by III. it is true when n is 2^r-1, and then, by III. again, when n is 2^r-2, and so on for each successive inferior integral value of n.

Similarly the theorem can be extended for *any* number of positive unequal numbers, when *m is between* 0 *and* 1.

EXAMPLES.—XIX.

1. Prove that $\{\underline{|n}\}^2 < \left\{\dfrac{(n+1)(2n+1)}{6}\right\}^n$;

and that $\{\underline{|n}\}^2 < \left\{\dfrac{n(n+1)^2}{4}\right\}^n$.

2. Show that $3m(3m+1)^2 > 4\sqrt[m]{\underline{|3m}}$.

3. If $a_1, a_2, \ldots a_n$ be n positive numbers greater than unity,

$$\frac{1}{n}\log(a_1 a_2 \ldots a_n) > (\log a_1 \log a_2 \log a_3 \ldots \log a_n)^{\frac{1}{n}}.$$

Why must they be greater than unity?

4. Prove that $\dfrac{1^4 + \ldots + n^4}{(n+1)^4} > \dfrac{n}{16}$.

5. Prove that $\underline{|2n-1} < \underline{|n}\,(2n)^{n-1}$, and that $(2x+a)\sqrt{a-x}$ takes its greatest real value when $x = \dfrac{a}{2}$.

6. Establish the inequality $\underline{|2n} < \underline{|n}\{\frac{4}{3}(4n^2-1)\}^{\frac{n}{2}}$.

MISCELLANEOUS EXAMPLES IN INEQUALITIES.—XX.

1. Prove that

$$2^{n(n+1)} > (n+1)^{n+1}\left(\frac{n}{1}\right)^n\left(\frac{n-1}{2}\right)^{n-1}\cdots\left(\frac{2}{n-1}\right)^2\frac{1}{n}.$$

2. In a G.P. the arithmetic mean of the extreme terms is greater than the arithmetic mean of the series.

3. Prove that $\dfrac{(ax)^{\frac{1}{2}} + (by)^{\frac{1}{2}}}{(a+b)^{\frac{1}{2}} + (x+y)^{\frac{1}{2}}}$ is a proper fraction.

4. Prove that $\{\underline{|n-1}\}^2 > n^{n-2}$.

5. If S be the sum of the mth powers, P the sum of the products m together, of the n numbers $a_1, a_2, \ldots a_n$, show that

$$\underline{|n-1}\, S > \underline{|n-m}\, \underline{|m}\, P.$$

6. If x, y, z be all positive, prove that

$$(x^2+y^2+z^2)^2(x+y+z)^2-8xyz(x^2+y^2+z^2)(x+y+z)-9x^2y^2z^2$$

is positive, unless $x=y=z$.

7. If x, y, z be such that any two of them are together greater than the third, $2(yz+zx+xy)>x^2+y^2+z^2$.

8. Show that $(a+b+c)^3>27abc<9(a^3+b^3+c^3)$, a, b, c being positive and not all equal.

9. If x, y, z be real numbers, prove that

$$a^2(x-y)(x-z)+b^2(y-z)(y-x)+c^2(z-x)(z-y)$$

will always be positive, provided that any two of the quantities a, b, c are together greater than the third.

10. If m is >1, and $x>1$, then $x^m-1>(x-1)^m$.

11. If n is positive, $(1+x)^n(1+x^n)>2^{n+1}x^n$.

12. Prove that $xyz>(y+z-x)(z+x-y)(x+y-z)$, each of the factors being positive.

13. Show that $n+1>2\sqrt[n]{\underline{|n}}$, n being a positive integer.

14. If a be >1, and m be any positive integer, show that $ma^m(a-1)>a^m-1$.

15. Prove that $\frac{a_1}{a_2}+\frac{a_2}{a_3}+\ldots+\frac{a_n}{a_1}>n$.

16. If n and m be positive integers,

(1.) $\frac{3n(n+1)}{4n+2}>\{\underline{|n}\}^{\frac{1}{n}}$;

(2.) $\left\{\frac{n+1}{2m-n+1}\right\}^n>\frac{\underline{|n}\,\underline{|m-n}}{\underline{|m}}$.

17. Show that $(1+a^p)^q>$ or $<(1+a^q)^p$, according as $q>$ or $<p$.

18. If $z^n=x^n+y^n$, prove that $z^m>$ or $<x^m+y^m$, according as $m>$ or $<n$.

19. Prove that $\frac{(r^n-1)^k}{r^{nk}-1}<\overline{n(r-1)}|^{k-1}$, if $k>1$ or <0.

XI

On the Roots and Coefficients of Equations.

112. We shall in this Chapter give a few interesting propositions relating to *positive integral* equations with *real* coefficients.

We assume that an equation of the nth degree has n roots.

It cannot have more than n different roots, for then a positive integral function of x would vanish for more than n different values of x, which it cannot do, by Art. 85, unless it vanishes for all values of x.

113. Prop. *To obtain the connexion between the coefficients of a positive integral function of the* n*th degree, such as*

$$p_n x^n + p_{n-1} x^{n-1} + \ . \ . \ . \ + p_1 x + p_0,$$

and the roots of the corresponding equation,

$$p_n x^n + p_{n-1} x^{n-1} + \ . \ . \ . \ + p_1 x + p_0 = 0 \quad . \quad (1)$$

Denote the function by $F(x)$, and let $a_1, a_2, \ . \ . \ . \ a_n$ be the roots of (1);

$\therefore$, App. to Pt. I. Art. 12, $p_n x^n + p_{n-1} x^{n-1} + \ . \ . \ . \ + p_1 x + p_0$

$$= p_n (x - a_1) \ . \ . \ . \ (x - a_n)$$

$$= p_n \{x^n - S_1 x^{n-1} + S_2 x^{n-2} + \ldots + (-1)^{n-1} S_{n-1} x + (-1)^n S_n\}$$

[Art. 413].

Equating coefficients we have,

$$p_{n-1} = -S_1 p_n, \qquad \therefore S_1 = -\frac{p_{n-1}}{p_n},$$

$$p_{n-2} = S_2 p_n, \qquad \therefore S_2 = \frac{p_{n-2}}{p_n},$$

etc. = etc.,

$$p_0 = (-1)^n S_n p_n, \ \therefore S_n = (-1)^n \frac{p_0}{p_n}.$$

Two simple cases of these results are given in [Art. 328].

114. *Ex.* 1. We have $a_1^2+a_2^2+\ldots+a_n^2$

$$=(a_1+a_2+\ldots+a_n)^2-2(a_1a_2+a_2a_3+a_1a_3+\text{etc.}),$$

$$=\left.\frac{p_{n-1}}{p_n}\right|^2-2\frac{p_{n-2}}{p_n}.$$

$$=\frac{1}{p_n^2}\{p_{n-1}^2-2p_{n-2}p_n\}.$$

Ex. 2. Determine the relation between p, q, r when the equation,
$$rx^3+x(2r-q)+p=0, \quad . \quad . \quad (1)$$
has two roots, of which the product $=-1$.

Let a, b, c denote the roots, so that $ab=-1$, . . (2).

Now by Art. 113, since there is no term in (1) involving x^2, we have
$$a+b+c=0, \quad . \quad . \quad . \quad (3),$$

$$\text{Also } bc+ca+ab=\frac{2r-q}{r} \quad . \quad . \quad (4),$$

$$abc=-\frac{p}{r} \quad . \quad . \quad (5);$$

$$\therefore \text{ from (2) and (5) } c=\frac{p}{r};$$

$$\therefore \text{ from (3) } a+b=-\frac{p}{r};\ \therefore c(a+b)=-\frac{p^2}{r^2};$$

$$\therefore \ \text{,,} \ (4)\ -\frac{p^2}{r^2}-1=\frac{2r-q}{r};$$

$$\therefore p^2+3r^2-rq=0.$$

EXAMPLES.—XXI.

1. If (x_1, y_1) (x_2, y_2) be the solutions of the equations, $y=m(x-a)$, $y^2=4ax$, then
$$x_1+x_2=2a+\frac{4a}{m^2},\ x_1x_2=a^2,\ y_1+y_2=\frac{4a}{m},\ y_1y_2=-4a^2.$$

2. Find the relation existing between p, q, r when the equation $x^3+px^2+qx+r=0$ has two roots equal to one another.

3. If a_1, a_2, a_3 be the roots of the equation $x^3+px^2+qx+r=0$, express, in terms of p, q, and r,

$$(1)\ \frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3};\quad (2)\ a_1^3+a_2^3+a_3^3;$$

$$(3)\ \frac{a_1}{a_2}+\frac{a_1}{a_3}+\frac{a_2}{a_3}+\frac{a_2}{a_1}+\frac{a_3}{a_1}+\frac{a_3}{a_2}.$$

4. Show by 2 that $x^3-5x^2+8x-4=0$ has two equal roots, and find all the roots.

5. If a, b, c, d be the roots of $x^4-x^2+x+1=0$, find the values of (1) $a^2b+a^2c+a^2d+b^2c+$etc.; (2) $a^3+b^3+c^3+d^3$.

6. If a, b, c be the roots of the equation $x^3+px^2+qx+r=0$, form the equations whose roots are (1) bc, ca, ab; (2) a^2, b^2, c^2; (3) $b-c$, $c-a$, $a-b$.

7. If α, β be the roots of the equation $ax^2+bx+c=0$, form the equation whose roots are $\alpha^2+\beta^2$, $\alpha^{-2}+\beta^{-2}$.

8. If the roots of the equation $x^3+px^2-3qx+r=0$ are in Harm. Prog. then $2q^3=r^2+pqr$.

9. Show that the roots of $x^3-3ax^2+b^2x+2a^3-ab^2=0$ are in Arith. Prog.

10. If the roots of $x^2+px+q=0$, and $x^2+qx+p=0$ differ by the same number, $p+q+4=0$.

11. Form the equation whose roots are the squares of the sum and difference of the roots of $2x^2+2(m+n)x+m^2+n^2=0$.

12. If the equations $x^4+px^3+qx^2+rx+1=0$, and $x^4+rx^3+qx^2+px+1=0$, have a common root, show that $p+r=q+2$, the symbols being essentially positive.

13. If two of the roots of the equation $x^3+qx+r=0$ are equal,

$$\frac{q^3}{4}+\frac{r^3}{27}=0.$$

14. If two of the roots of the equation $ax^3+3bx^2+3cx+d=0$ are equal, $4(ac-b^2)(bd-c^2)-(ad-bc)^2=0$.

15. The equation $x^3-8x^2-21x+r=0$ has one root 5; find the other roots and the value of r.

115. Prop. *In such equations as we are considering, imaginary roots enter by pairs.*

In the expression $p_n x^n + p_{n-1}x^{n-1} + \ldots + p_1 x + p_0$, if we substitute $\alpha + \beta\sqrt{-1}$ for x, each term can be put into the form $A + B\sqrt{-1}$ (Art. 23), and therefore the whole expression can be put into this form, say $P + Q\sqrt{-1}$.

Hence if $\alpha + \beta\sqrt{-1}$ is a root of the corresponding equation, $P + Q\sqrt{-1} = 0$, $\therefore$ $P = 0$ and $Q = 0$.

Now if we had substituted $\alpha - \beta\sqrt{-1}$, instead of $\alpha + \beta\sqrt{-1}$, the only effect would have been to change the sign of each term involving an odd power of $\sqrt{-1}$, and therefore the expression would reduce to $P - Q\sqrt{-1}$, which vanishes, since P and Q vanish; $\therefore$ $\alpha - \beta\sqrt{-1}$ is a root of the equation.

Obs. The imaginary expressions $\alpha \pm \beta\sqrt{-1}$ are said to be conjugate to each other.

116. In the same way it can be shown, that if $a + \sqrt{b}$ is a root of an equation, involving positive integral powers of x with real and rational coefficients, then $a - \sqrt{b}$ is so also, b not being a perfect square. Also if $a + \sqrt{b} + \sqrt{c}$ is a root, c not being a perfect square, then $a - \sqrt{b} + \sqrt{c}$, $a - \sqrt{b} - \sqrt{c}$, $a + \sqrt{b} - \sqrt{c}$ are also roots of the same equation.

117. *Ex.* Given that $3 + \sqrt{2}$ is a root of the equation,

$$x^3 - 11x^2 + 37x - 35 = 0,$$

find the rest.

By Art. 116, $3 - \sqrt{2}$ is another root; let y denote the third;

$$\therefore y(3 + \sqrt{2})(3 - \sqrt{2}) = 35, \qquad \text{Art. 113};$$

$$\therefore y = \frac{35}{7} = 5.$$

118. From Art. 115, 116 it is evident that a cubic equation must have one real, rational root; for it cannot have more than three roots, and it must have no imaginary roots, or else an even number, therefore it can only have two. Similarly we see that it must have no irrational roots, or else two.

In the same way we can see that any equation of odd degree must have one real rational root.

EXAMPLES.—XXII.

1. The equation $x^3+3x^2+qx-13=0$, has one root $=-2+3\sqrt{-1}$, find the rest, and the value of q.

2. One root of $x^4-3x^2-42x+40=0$ is $\frac{1}{2}(-3+\sqrt{-31})$, find the others.

3. Solve the equation $x^4+2x^3-4x^2-4x+4=0$, having given that one root is $\sqrt{2}$.

4. One root of the equation $x^4-4x^3-8x+32=0$ is $-1+\sqrt{-3}$, find the rest.

5. Form the equation of the 4th degree, of which one root is $\frac{1}{2}\sqrt{3}+\sqrt{-1}$.

6. Form the cubic equation having 5 and $-\frac{1}{2}(5+3\sqrt{-3})$ for two of its roots.

7. If $a\pm\beta\sqrt{-1}$ be the imaginary roots of $x^3+qx+r=0$, then $\beta^2=3a^2+q$.

119. Prop. *To find the cube roots of unity.*

This is the same as finding the roots of the equation

$$x^3-1=0 \quad . \quad . \quad . \quad (1).$$

Evidently 1 is a root of (1).

Now $x^3-1=(x-1)(x^2+x+1)$;

$\therefore$ the other two roots are given by $x^2+x+1=0$,

i.e. they are $\dfrac{-1\pm\sqrt{-3}}{2}$.

Obs. Let

$$\omega=\frac{-1+\sqrt{-3}}{2}, \text{ then } \omega^2=\left(\frac{-1+\sqrt{-3}}{2}\right)^2=\frac{1-2\sqrt{-3}-3}{4}$$

$$=\frac{-1-\sqrt{-3}}{2}, \text{ the other root.}$$

Similarly, if we put $\omega=\dfrac{-1-\sqrt{-3}}{2}$, then the other root is ω^2.

Also ω^3 and ω^6 are of course both equal to 1.

120. *Ex.* Since $x+y+z$ is a factor of $x^3+y^3+z^3-3xyz$; $\therefore$ $x+\omega y+\omega^2 z$ is a factor of $x^3+\overline{\omega y}|^3+\overline{\omega^2 z}|^3-3x.\omega y.\omega^2 z$; but if ω, ω^2 are the two imaginary roots of unity, this last expression $=x^3+y^3+z^3-3xyz$; $\therefore$ $x+\omega y+\omega^2 z$ is a factor of $x^3+y^3+z^3-3xyz$.

Similarly it can be shown that $x+\omega^2 y+\omega z$ is a factor of it;
$\therefore$ $x^3+y^3+z^3-3xyz=N(x+y+z)(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z)$, and $N=1$, as we see by equating the coefficients of x^3.

EXAMPLES.—XXIII.

[In these examples ω denotes an imaginary cube root of unity.]

1. Prove that

   (1) $\omega^2+\omega+1=0$, and that ω^{-1} is also a cube root of unity;

   (2) $(1+\omega)^{3r}=\pm 1$;

   (3) $(1+\omega)^{3r-1}=\pm\,\omega$; r being any integer.

2. Find the cube roots of -1.

3. Prove that the cube roots of 8 are 2, 2ω, $2\omega^2$.

4. What are the cube roots of -8?

5. If a be a cube root of A, prove that $a\omega$, $a\omega^2$ are the other two cube roots.

XII

Partial Fractions.

121. If $\dfrac{3x^2+1}{(x+1)(x^2-x+1)}=\dfrac{A}{x+1}+\dfrac{Bx+C}{x^2-x+1}$, for all values of x; required to find the values of A, B, C.

Clearing of fractions we have

$$\begin{aligned}3x^2+1&=A(x^2-x+1)+(Bx+C)(x+1)\\&=(A+B)x^2+x(B+C-A)+A+C \text{ for all values of } x;\end{aligned}$$

$\therefore$, equating coefficients,

$$\begin{aligned}A+B\quad&=3,\\B+C-A&=0,\\A+C\quad&=1;\end{aligned}$$

$$\therefore A=\frac{4}{3},\quad B=\frac{5}{3},\quad C=-\frac{1}{3}.$$

Hence $\dfrac{3x^2+1}{x^3+1}=\dfrac{4}{3(x+1)}+\dfrac{5x-1}{3(x^2-x+1)}$.

Each of the fractions, into which $\dfrac{3x^2+1}{x^3+1}$ has been *resolved*, is called a partial, or simple, fraction.

122. It is proved in works on the Integral Calculus that any fraction can be resolved into partial fractions, in the manner indicated below,

(1) $\dfrac{3x^2+4x+2}{x(x-1)(x-2)^2}=\dfrac{A}{x}+\dfrac{B}{x-1}+\dfrac{C}{x-2}+\dfrac{D}{(x-2)^2}$,

(2) $\dfrac{2x-5}{(x-3)^2(x^2+x+1)}=\dfrac{A}{x-3}+\dfrac{B}{(x-3)^2}+\dfrac{Cx+D}{x^2+x+1}$.

(3) $\dfrac{7x^6+3x-2}{(x-1)^2(x^2-x+2)(x^2+x-1)^2}=\dfrac{A}{x-1}+\dfrac{B}{(x-1)^2}+\dfrac{Cx+D}{x^2-x+2}$
$$+\frac{Ex+F}{x^2+x-1}+\frac{Gx+H}{(x^2+x-1)^2}.$$

We thus decompose a fraction into a number of fractions, such that the L. C. M. of their denominators is the denominator of the given fraction. We must therefore find all the real factors of which this denominator is the product, and then to each rational factor will correspond one, or more, partial fractions, according as the factor is raised to the first, or some higher power.

It only remains to determine the values of the coefficients occurring in the numerators. This can be done by clearing of fractions, then since the equality we thus obtain exists for all values of x, we may equate the coefficients of the like powers of x on each side, and produce a sufficient number of equations for the determination required.

123. If in the fraction to be decomposed the numerator is of the same dimensions as the denominator, or higher, we must first transform it into an expression involving a fraction in which the numerator is of lower dimensions than the denominator.

Thus $\dfrac{3x^5+4x^3+2x+1}{x^3-8}=3x^2+4+\dfrac{24x^2+2x+33}{x^3-8}$,
$$=3x^2+4+\frac{A}{x-2}+\frac{Bx+C}{x^2+2x+4};$$
$$\therefore\ 24x^2+2x+33=A(x^2+2x+4)+(Bx+C)(x-2);$$
$$\therefore\ A+B\quad=24,$$
$$2A+C-2B=2,$$
$$4A-2C\quad=33.$$

Hence $A=\dfrac{133}{12}$, $B=\dfrac{155}{12}$, $C=\dfrac{68}{12}$;
$$\therefore\ \frac{3x^5+4x^3+2x+1}{x^3-1}=3x^2+4+\frac{133}{12(x-2)}+\frac{155x+68}{12(x^2+2x+4)}.$$

124. Instead of determining the numerators by solving equations obtained from equating coefficients, it is often more convenient to adopt a method of which the following is an example:—

$$\frac{2x^3+x-1}{(x-2)(x^2+x+1)^2}=\frac{A}{x-2}+\frac{Bx+C}{x^2+x+1}+\frac{Dx+E}{(x^2+x+1)^2};$$

$$\therefore\ A(x^2+x+1)^2+(Bx+C)(x-2)(x^2+x+1)+(Dx+E)(x-2)$$
$$=2x^3+x-1 \quad . \quad . \quad . \quad (1).$$

Put $x^2+x+1=0$, and $\therefore\ x^2=-x-1$, . (2);

$$\therefore\ (Dx+E)(x-2)=2x^3+x-1=2x(-x-1)+x-1;$$
$$\therefore\ Dx^2+(E-2D)x-2E=-2x^2-x-1;$$
$$\therefore\ D(-x-1)+(E-2D)x-2E=-2(-x-1)-x-1;$$
$$\therefore\ (E-3D)x-D-2E=x+1 \quad . \quad . \quad (3);$$

but from (2) $x=\dfrac{-1\pm\sqrt{-3}}{2}$;

$$\therefore\ (E-3D)\left(-\frac{1}{2}\pm\frac{\sqrt{-3}}{2}\right)-D-2E=\left(-\frac{1}{2}\pm\frac{\sqrt{-3}}{2}\right)+1.$$

Equating imaginary parts, $E-3D=1$, and $\therefore\ -D-2E=1$, which result we might have obtained by equating coefficients in (3).

$$\text{Hence } D=-\frac{3}{7},\quad E=-\frac{2}{7}.$$

Now in (1), bring the terms containing D and E to the right-hand side, multiply throughout by 7, put $7D=-3$, $7E=-2$, and divide by x^2+x+1. We obtain

$$7A(x^2+x+1)+7(Bx+C)(x-2)$$
$$=\frac{14x^3+7x-7+(3x+2)(x-2)}{x^2+x+1}$$
$$=\frac{14x^3+3x^2+3x-11}{x^2+x+1};$$

$$\therefore\ 7A(x^2+x+1)+7(Bx+C)(x-2)=14x-11 \quad . \quad . \quad (4).$$

Again, if we treat (4) as we did (1), *i.e.* put $x^2=-x-1$, and equate coefficients when we have reduced it to an equation similar to (3), we shall obtain

$$B=-\frac{17}{49},\quad C=\frac{47}{49}.$$

Now in (4), bring the terms containing B and C to the right-hand side, multiply throughout by 7, put $49B=-17$, $49C=47$, and divide by x^2+x+1. We obtain

$$49A=\frac{98x-77+(17x-47)(x-2)}{x^2+x+1}=17;$$

$$\therefore A=\frac{17}{49}.$$

After we had found B and C we might have obtained A at once from (4), by equating coefficients of x^4, thus

$$7A+7B=0; \quad \therefore A=-B=\frac{17}{49}.$$

Or we might have begun with finding A.

Thus, put $x=2$ in (1), we obtain $49A=17$.

Now in (1) bring the term involving A to the right side, multiply throughout by 49, put $49A=17$, and divide by $x-2$.

We should have then continued, by finding D and E, etc., as before.

For De Morgan's proof of the existence of partial fractions corresponding to rational factors, see Todhunter's Integral Calculus, and Sect. VIII. of Homersham Cox's Integral Calculus, published in *Weale's Rudimentary Series.*

125. *Ex.* Expand $\frac{4+2x}{2+5x-3x^2}$ into a series of ascending powers of x.

We have
$$\frac{4+2x}{2+5x-3x^2}=\frac{2}{2-x}+\frac{1}{1+3x}$$
$$=\left(1-\frac{x}{2}\right)^{-1}+(1+3x)^{-1}$$
$$=2-\frac{5}{2}x+\text{ etc.}$$

the coefficient of x^n being $\frac{1}{2^n}+(-1)^n.3^n$.

EXAMPLES.—XXIV.

[It is to be noted that the words Resolve, Decompose, Separate, Reduce are here synomymous.]

1. Supposing the equation

$$\frac{a}{x+1}+\frac{b}{x+2}+\frac{c}{x+3}=\frac{x^2}{(x+1)(x+2)(x+3)}$$

to be true for all values of x, investigate the values of a, b, c.

2. Reduce to their partial fractions the following :—

(1) $\dfrac{x+2}{x^2-5x+6}$, (2) $\dfrac{x^2-x+1}{(x^2+1)(x-1)^2}$, (3) $\dfrac{5+6x-2x^2}{(3+2x)^2}$,

(4) $\dfrac{3x}{x^2+7x+6}$, (5) $\dfrac{Ax+B}{(x^2+a^2)(x^2+b^2)}$, (6) $\dfrac{x^2-7x+1}{x^3+6x^2+11x+6}$,

(7) $\dfrac{x^2+x+1}{(x-1)(x-2)(x-3)}$.

3. Resolve the following into their partial fractions :—

(1) $\dfrac{7x}{(2x-3)(x+2)^2}$, (2) $\dfrac{11x^2-26x+107}{x^3-x^2-21x+45}$,

(3) $\dfrac{3x^3-2x^2+4x-3}{(x^2-x+1)(x^2-1)}$, (4) $\dfrac{x^2}{(x+1)^3(x-1)^2}$,

(5) $\dfrac{x^2-x+7}{x^3-3x^2+4}$, (6) $\dfrac{5x^2+3x+1}{(x^2+7x+5)(x+2)^2}$,

(7) $\dfrac{2x^5+2x^4+6x^2-8x+4}{x^4+x^3-2x^2}$.

4. Separate $\dfrac{4x-1}{(x+2)(x^2+5)^2}$ into partial fractions.

5. Decompose $\dfrac{5-10x}{2-x-3x^2}$ into partial fractions; and expand each fraction in powers of x, giving the general term.

6. Expand the following into series ascending by powers of x:—

(1) $\dfrac{5x^3+1}{x^3-3x+2}$, (2) $\dfrac{4-3x}{2-x-3x^2}$,

(3) $\dfrac{3x-2}{(x-1)(x-2)(x-3)}$, (4) $\dfrac{1}{1-3x+2x^2}$.

7. Expand $\dfrac{ab}{(a-x)(b-x)}$ in a series ascending by powers of x as far as x^3; also prove that the coefficient of x^r is $\dfrac{b^{r+1}-a^{r+1}}{a^r b^r(b-a)}$, and show what this becomes when $a=b$.

8. Find the coefficient of $x^m y^n$ in the expansion of

$$\frac{x(1-ax)}{(1-x)(1-ax-by)}.$$

9. Find two fractions whose difference is $\dfrac{x}{x^2-5x+6}$.

XIII

Interest, etc.

126. Suppose that a debt lasts for $\overline{n+p}$ of the equal intervals, at the end of each of which interest is due for that interval, n being a whole number and p a proper fraction.

Let r be the ratio which interest bears to its principal for one of these intervals, so that if F be the principal during one interval Fr is the interest due at the end of it.

Thus if interest is due every month at the rate of 5 per cent. per annum, the interest on £100 for a month is £$\frac{5}{12}$, and therefore $r = \frac{5}{12} : 100 = \frac{1}{240}$.

127. Let P denote the principal of the debt at first.

A „ its amount at the end of the $\overline{n+p}$ intervals.

Then with *compound interest*, as in [Art. 472], the amount at the end of the n intervals is $P(1+r)^n$;

$\therefore$ the interest for the next interval$=rP(1+r)^n$,

and „ „ pth part of it$=prP(1+r)^n$.

Hence $A = P(1+r)^n + prP(1+r)^n = P(1+r)^n(1+pr)$ (1).

Also with *simple interest*, since the interest due at the end of each interval is reckoned on the original principal only, Pr is the interest for each of the n intervals, and pPr for the pth part at the end.

Hence $A = P + nPr + pPr = P\{1 + \overline{n+p}r\}$ . . (2).

Obs. 1. From (1) and (2) it is evident that any formula for simple interest can be deduced from the corresponding one for compound, by neglecting r^2 and all higher powers. Therefore

in general theorems we shall confine our attention to compound interest.

Obs. 2. The form of the formula for simple interest is the same, whether the time is a whole number of intervals or not, since n and p appear, not separately, but combined in $n+p$.

128. A debt, if allowed to remain with its interest unpaid, increases in value as time goes on, and conversely, a debt may be equitably discharged at any date before it is due by paying such a sum as will, with its interest for the interval, amount to the value of the debt when due.

This sum is the *value of the debt* at the *date of the supposed discharge;* its interest is called the *discount* on the debt for the interval, and is evidently the difference between the values of the debt when due and at the date of discharge.

129. A debt, whose value when due is A, is called a *debt* A.

130. By the *present worth* of a debt due some time hence is meant its value at the present time.

By Art. 127 we see that, if there is a debt A, its value $\overline{n+p}$ intervals before it is due is $\frac{A}{(1+r)^n(1+pr)}$, and $\therefore$ the discount on it is $A-\frac{A}{(1+r)^n(1+pr)}$.

The value, s intervals hence, of a debt A, due n intervals hence, is of course its value $n-s$ intervals before it is due, and is therefore represented by $\frac{A}{(1+r)^{n-s}}$.

If $s>n$, $n-s$ is negative, then $\frac{A}{(1+r)^{n-s}}$, or $A(1+r)^{s-n}$, represents the value of the debt $s-n$ intervals *after* it is due.

131. The application of logarithms, as in [Art. 473], to the formulæ of Art. 130 would be tedious, and instead $\frac{A}{(1+r)^n(1+pr)}$ is generally reduced to a simple form by one, *or other, of* the two following methods of approximation :—

(1) We neglect r^2 and higher powers in the denominator, which gives $\frac{A}{1+(n+p)r}$ for the present worth, and $\frac{(n+p)r}{1+(n+p)r}$ for the discount.

This is equivalent to working with simple, instead of compound, interest, and is the method usually given in ordinary arithmetic.

(2) We proceed as before to obtain $\frac{A}{1+(n+p)r}$, which we then expand in a series, and neglect r^2 and higher powers, which gives $A\{1-(n+p)r\}$ for the present worth, and $A(n+p)r$ for the discount.

This is equivalent to reckoning discount as the simple interest on the value of the debt, *when due*, instead of $n+p$ intervals before.

Discount so reckoned is called bankers' discount.

132. The present value of a debt of £1 due n years hence is represented by $(1+r)^{-n}$. Tables have been made of the values of this expression, for all integral values of n from 1 to 100, and for different rates of interest between 2 per cent. and 10 per cent. See Jones's "Treatise on Annuities," published in the *Library of Useful Knowledge.*

133. If r be the ratio of interest to principal when the former is due yearly, $\frac{r}{m}$ is the ratio when it is due at every mth part of a year.

Now if on being due it immediately begins to bear interest itself, the amount (A) in n years time is given by

$$A=P\left(1+\frac{r}{m}\right)^{mn}=P\left\{\left(1+\frac{r}{m}\right)^{\frac{m}{r}}\right\}^{nr}.$$

Hence, if m is made indefinitely large, *i.e.*, if the interest is due every instant, we have $A=Pe^{nr}$.

134. B *has to pay* C £a, £b, . . . £s *at the ends of* $a, \beta, \ldots \sigma$ *years respectively. Find the sum* £x *which he must pay at the end of* ξ *years, in order to be free of the debt, interest being paid annually.*

By Art. 130 the values of the debts ξ years hence will be respectively,

$$a(1+r)^{\xi-a},\ b(1+r)^{\xi-\beta},\ \ldots\ldots\ s(1+r)^{\xi-\sigma};$$
$$\therefore\ x=a(1+r)^{\xi-a}+b(1+r)^{\xi-\beta}+\ \ldots\ +s(1+r)^{\xi-\sigma};$$
$$\therefore\ x(1+r)^{-\xi}=a(1+r)^{-a}+\ \ldots\ldots\ +s(1+r)^{-\sigma}.$$

If we make the 2nd approximation of Art. 131, we obtain

$$x(1-\xi r)=a(1-ar)\ \ldots\ldots\ldots\ +s(1-\sigma r);$$

and now by Art. 127, *Obs.* 2, it is immaterial whether $\xi, a, \ldots \sigma$ represent whole numbers of years or not.

If $x=a+b+\ \ldots\ +s$, we have $x\xi=aa+b\beta+\ \ldots\ +s\sigma$, which is the expression of the rule for the *equation of payments*, and is independent of the rate of interest.

The name of *Equated Time* has been given to ξ.

135. B *makes* C n *annual payments of* £P *each. What sum will* C *owe* B *immediately after the last payment, interest being due once a year?*

At the stated time C will owe B

	$P(1+r)^{n-1}$	on account of the	1st payment,
	$P(1+r)^{n-2}$	"	2nd "
	etc.,	etc.,	
and	P	"	last "

$$\therefore \text{ he will owe } B \text{ altogether } P\frac{(1+r)^n-1}{r}.$$

In the same way, if an annual payment of £P has been due to B, but has not been paid for n years, there is owing to him

$$£P\frac{(1+r)^n-1}{r},$$

immediately after the nth payment should have been made.

Any such annual payment is called an Annuity. In the last case the annuity is said to be *forborne*; and $P\dfrac{(1+r)^n-1}{r}$ is thus the value of an Annuity of £P forborne for n years.

136. *What sum must* C *pay to* B, *in order that he may receive from* B *an annual payment of* £P *for* n *years, the first being received a year after he made the payment to* B?

Here $\dfrac{P}{1+r}$ will amount to P in one year.

$\dfrac{P}{(1+r)^2}$ " " two years.

. . . .

$\dfrac{P}{(1+r)^n}$ " " n "

Hence C must pay

$$\frac{P}{1+r}+\frac{P}{(1+r)^2}+\ \cdots\ +\frac{P}{(1+r)^n}$$
$$=P\frac{(1+r)^n-1}{r(1+r)^n}=\frac{P}{r}\left\{1-\frac{1}{(1+r)^n}\right\}.$$

The sum that C has to pay for the annuity of £P is called the present value of the annuity.

Of course the results in this, and the preceding Articles, may be modified by the two approximations mentioned in Art. 127.

EXAMPLES.—XXV.

1. The simple interest on a certain sum of money for a certain time is £7, and the discount for the same time at the same rate of simple interest is £6. What is the sum of money? If the time be $3\frac{1}{2}$ years, what is the rate per cent.?

2. A man has £20,000 and makes 4 per cent. of his capital. He spends £1000 a year. How long before he is bankrupt, supposing that at the beginning of each year, during which any £1000 is spent, it is withdrawn from bearing interest?

3. Prove that discount is half the harmonic mean between the principal and interest.

The interest on a certain sum of money is £180, and the discount on the same sum for the same time and at the same rate of interest is £150. Find the sum.

4. What is the present worth of the reversion of a freehold estate of £882 per annum to commence two years hence, allowing interest at 5 per cent.?

5. If the present value of a perpetual annuity be twenty-five times that of the annuity, what is the rate of interest?

6. Find the amount, at compound interest, of P pounds for n years at a given rate per cent., the interest being paid q times a year; and show that the interest thus obtained is to the interest of the same at simple interest in the ratio of $2q+(qn-1)r : 2q$ nearly, where r is the interest of one pound for one year.

7. Two men invest sums of £4410 and £4400 respectively, at the same rate of interest, the former at simple, the latter at compound interest; and at the end of two years their properties amount to equal sums. Find the rate of interest.

8. I borrow £1000 on condition that I repay £10 at the end of every month for 10 years. Find an equation which will determine the rate of interest I pay.

9. What is the present value of an annuity of £100 to commence after ten years, reckoning 3 per cent. compound interest? If each payment is to be m times the preceding, within what limits must m lie in order that the present value may be finite?

10. If the interest on £A for a year be equal to the discount on £B for the same time, find the rate of interest.

11. If the three per cents. are at 90 one month before the payment of the half-yearly dividend, what is the rate of interest?

Find the value of a perpetual annuity of £A payable half-yearly, to commence one month hence, reckoning compound interest payable half-yearly.

12. What annuity is equivalent to a sum of £200 paid at the

end of every two years, the rate of interest being 5 per cent. per annum? Show from general reasoning that it is less than £100.

13. A person lends at the end of one year £1, at the end of two years £4, at the end of three years £9, etc., at simple interest. What will the debt amount to at the end of n years?

14. If P be the present value of an annuity to continue for p years, and $P+Q$ for $2p$ years, the annuity

$$=\frac{P^2}{P-Q}\left\{\left(\frac{P}{Q}\right)^{\frac{1}{p}}-1\right\}.$$

15. Find the present value of an annuity of £1, paid n times per annum, and continuing for m years, allowing compound interest at the rate of r per cent. per annum; and prove that, as n is indefinitely increased, this present value continually approaches the limit $\dfrac{1-e^{-mr}}{r}$.

16. A has just purchased an annuity for ever, and B, with the same capital, one for three years, when an income tax for three years is imposed. If the tax be 3 per cent. on the perpetual annuity, what ought it to be on B's annuity, if the value of both properties be taxed alike? (Allow compound interest of 5 per cent. per annum.)

17. Compare the present values of two scholarships, payable half-yearly, one of £50 to continue for two and a half years, and the other of £35 to continue for four and a half, reckoning 5 per cent. simple interest.

18. If p years' purchase must be paid for an annuity to continue a certain number of years, and q years purchase for an annuity to continue twice as long, determine the rate per cent., supposing the annuity to commence a year after purchase in each case.

XIV

Continued Fractions.

137. Any such fraction as $3+\cfrac{5}{7+\cfrac{6}{8+\cfrac{3}{4}}}$, of which the general form is $a_1+\cfrac{b_2}{a_2+\cfrac{b_3}{a_3+\cfrac{b_4}{a_4+\dots}}}$, is called a continued fraction.

For the sake of economizing space, such a fraction is written thus

$$a_1+\frac{b_2}{a_2+}\,\frac{b_3}{a_3+}\,\frac{b_4}{a_4+}\text{ etc.}$$

138. To simplify the numerical fraction given above, we proceed thus;

$$8+\frac{3}{4}=\frac{35}{4};\ \therefore\ \frac{6}{8+\frac{3}{4}}=\frac{6}{\frac{35}{4}}=\frac{24}{35};$$

$$\therefore\ 7+\frac{6}{8+\frac{3}{4}}=7+\frac{24}{35}=\frac{245+24}{35}=\frac{269}{35};$$

$$\therefore\ \frac{5}{7+}\,\frac{6}{8+}\,\frac{3}{4}=\frac{5}{\frac{269}{35}}=\frac{175}{269};$$

$$\therefore\ 3+\frac{5}{7+}\,\frac{6}{8+}\,\frac{3}{4}=\frac{807+175}{269}=\frac{982}{269}.$$

See Hamblin Smith's *Arithmetic*, 2nd edition, Art. 76.

139. In the same way by taking *any continued fraction*, which is *not endlessly* prolonged, we can show that its *value can be expressed by a vulgar fraction having its numerator and denominator whole numbers*, *i.e.*, such a continued fraction always represents a rational number.

140. Hence, since a surd cannot be a rational number, we cannot put it into the form of a terminating continued fraction.

141. We shall for the most part hereafter consider only those continued fractions in which $b_2 = b_3 = \text{etc.} = 1$, that is, only those of the form

$$a_1 + \frac{1}{a_2+}\,\frac{1}{a_3+}\ \text{etc.}$$

142. Prop. *To express the fraction* $\frac{A}{B}$ *in the form of a continued fraction.*

Divide A by B, let a_1 be quotient and C remainder;

$$\therefore \frac{A}{B} = a_1 + \frac{C}{B} = a_1 + \frac{1}{\frac{B}{C}}.$$

Again divide B by C, let a_2 be quotient and D remainder;

$$\therefore \frac{B}{C} = a_2 + \frac{D}{C} = a_2 + \frac{1}{\frac{C}{D}};$$

$$\therefore \frac{A}{B} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{\frac{C}{D}}}.$$

By continuing this process as far as possible we express $\frac{A}{B}$ in the form of a continued fraction.

Obs. 1. It will be observed that the process here pursued is identical with that for finding the H. C. F. of A and B; hence if A and B be two rational numbers we shall at some point or other come to a division having no remainder, and then the process terminates, and we obtain a terminating continued fraction.

Hence *any* vulgar fraction can be put into the form of a terminating continued fraction.

Obs. 2. If $\dfrac{A}{B}$ is a proper fraction, $a_1=0$.

Obs. 3. Since $\dfrac{A}{B}=a_1+$a fraction, $a_1<\dfrac{A}{B}$.

143. Prop. *To express a quadratic surd in the form of a continued fraction.*

We will exemplify the method by converting $\sqrt{13}$ into a continued fraction.

$$\sqrt{13}=3+\sqrt{13}-3=3+\frac{(\sqrt{13}-3)(\sqrt{13}+3)}{\sqrt{13}+3}=3+\frac{4}{\sqrt{13}+3}$$

$$=3+\cfrac{1}{\cfrac{\sqrt{13}+3}{4}},$$

$$\frac{\sqrt{13}+3}{4}=1+\frac{\sqrt{13}-1}{4}=1+\frac{3}{\sqrt{13}+1}=1+\cfrac{1}{\cfrac{\sqrt{13}+1}{3}},$$

$$\frac{\sqrt{13}+1}{3}=1+\frac{\sqrt{13}-2}{3}=1+\frac{3}{\sqrt{13}+2}=1+\cfrac{1}{\cfrac{\sqrt{13}+2}{3}},$$

$$\frac{\sqrt{13}+2}{3}=1+\frac{\sqrt{13}-1}{3}=1+\frac{4}{\sqrt{13}+1}=1+\cfrac{1}{\cfrac{\sqrt{13}+1}{4}},$$

$$\frac{\sqrt{13}+1}{4}=1+\frac{\sqrt{13}-3}{4}=1+\frac{1}{\sqrt{13}+3},$$

$$\sqrt{13}+3=6+\sqrt{13}-3.$$

After this, since we have a second time come to the expression $\sqrt{13}-3$, the figures will be identically repeated.

Hence $\sqrt{13}=3+\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{6+}\,\frac{1}{1+}$ etc.

Note.—The process performed in each line is the same; for example, consider the third line, we first seek the greatest integer in $\frac{\sqrt{13}+1}{3}$, it is 1; $\therefore\ \frac{\sqrt{13}+1}{3}-1$, or $\frac{\sqrt{13}-2}{3}$, is a fraction, and the rest of the line is taken up with rationalizing the numerator of this fraction, so that when we invert it we may have the expression $\frac{1}{\frac{\sqrt{13}+2}{3}}$, in which the denominator consists of a fraction having a rational denominator.

EXAMPLES.—XXVI.

Express as continued fractions

1. $\frac{167}{81}$. 2. $\frac{81}{167}$. 3. $2\frac{39}{73}$. 4. $\cdot 23$. 5. $\cdot 0053$.

6. $1\cdot 029$. 7. $\sqrt{3}$. 8. $\sqrt{10}$. 9. $\sqrt{8}$. 10. $\sqrt{7}$.

11. $\sqrt{17}$. 12. $\sqrt{32}$. 13. $\sqrt{45}$. 14. $\sqrt{44}$. 15. $3\sqrt{11}$.

16. $5+3\sqrt{3}$. 17. $\frac{1}{\sqrt{19}}$. 18. $\frac{1}{\sqrt{23}}$. 19. $\frac{86400}{20929}$.

20. Show that

$$na_1+\frac{1}{na_2+}\,\frac{1}{na_3+}\text{ etc.}=n\left\{a_1+\frac{1}{n^2a_2+}\,\frac{1}{a_3+}\,\frac{1}{n^2a_4+}\text{ etc.}\right\}.$$

144. *Def.* The symbols $a_1, a_2, \ldots a_r, \ldots$ are called the first, second, ... rth, ... *partial quotients*, or simply quotients.

The part consisting of a_r and all that follows it, is called the rth complete quotient.

Thus any partial quotient is the integral part of the corresponding complete quotient.

The part $a_1 + \frac{1}{a_2+}\frac{1}{a_3+}$ etc. ... $\frac{1}{a_r}$ is called the rth *convergent* to the continued fraction, or simply, the rth convergent.

Thus each convergent forms a terminating continued fraction, and we could of course find its value in the same way as in Art. 138, by beginning at the bottom, but this would be laborious; we proceed therefore to establish a rule, by which the value of each convergent after the second may be deduced from the values of the two preceding.

145. PROP. *The numerator of the* n*th convergent is equal to the product of the* n*th quotient and the numerator of the* (n−1)*th convergent, increased by the numerator of the* (n−2)*th convergent, and the same is true,* mutatis mutandis, *for the denominator,* n *being greater than* 2.

That is to say if $\frac{p_n}{q_n}$ denote the nth convergent, then

$$p_n = a_n p_{n-1} + p_{n-2},$$

$$q_n = a_n q_{n-1} + q_{n-2}.$$

The first convergent $a_1 = \frac{a_1}{1}$.

The second $a_1 + \frac{1}{a_2} = \frac{a_1 a_2 + 1}{a_2}$.

The third $a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3}} = a_1 + \frac{a_3}{a_2 a_3 + 1} = \frac{a_3(a_1 a_2 + 1) + a_1}{a_3 a_2 + 1}$.

Hence the rule holds for the 3rd convergent.

Suppose it holds for any one, say the rth. Now the $(r+1)$th differs from the rth only in having the more complete quotient $a_r + \dfrac{1}{a_{r+1}}$ instead of a_r;

$$\therefore \quad \frac{p_{r+1}}{q_{r+1}} = \frac{\left(a_r + \dfrac{1}{a_{r+1}}\right) p_{r-1} + p_{r-2}}{\left(a_r + \dfrac{1}{a_{r+1}}\right) q_{r-1} + q_{r-2}}$$

$$= \frac{a_{r+1}(a_r p_{r-1} + p_{r-2}) + p_{r-1}}{a_{r+1}(a_r q_{r-1} + q_{r-2}) + q_{r-1}}$$

$$= \frac{a_{r+1} p_r + p_{r-1}}{a_{r+1} q_r + q_{r-1}};$$

$\therefore$, if the rule holds for any one convergent, it holds for the succeeding one; but it *does* hold for the 3rd, $\therefore$ for the 4th, and so on generally.

COR. The convergents form a series of fractions, in which each numerator is greater than any preceding numerator, and each denominator is greater than any preceding denominator.

EXAMPLES.—XXVII.

1. Find the quotients and convergents obtained in converting into continued fractions the following numbers :—

(1) $\frac{92}{73}$, (2) $9\frac{3}{8}$, (3) $\frac{3}{11}$, (4) $\frac{17}{9}$.

2. Find the first six quotients and convergents of the continued fractions corresponding to the following numbers :—

(1) $\sqrt{5}$, (2) $\sqrt{14}$, (3) $2\sqrt{5}$, (4) $3+5\sqrt{3}$.

3. How many quotients are there obtained in converting $\dfrac{31}{9}$ into a continued fraction. Find the difference between each successive convergent and the whole fraction.

4. Determine the values of the fractions,

(1) $2 + \dfrac{1}{3+}\,\dfrac{1}{4+}\,\dfrac{1}{2+}\,\dfrac{1}{3}$, and (2) $\dfrac{1}{3+}\,\dfrac{1}{4+}\,\dfrac{1}{2+}\,\dfrac{1}{3+}\,\dfrac{1}{2}$;

also the differences between those values and the successive convergents to the fractions.

5. Find the difference between each consecutive pair of convergents to the continued fraction

$$3+\frac{1}{5+}\,\frac{1}{7+}\,\frac{1}{9+}\,\frac{1}{2}.$$

6. If $\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3}$ be successive convergents to a continued fraction, $\frac{p_3-p_1}{q_3-q_1}=\frac{p_2}{q_2}$.

7. If $\frac{a_n}{b_n}$ be the nth convergent of a continued fraction, prove that $\left(\frac{a_{n+2}}{a_n}-1\right)\left(1-\frac{a_{n-1}}{a_{n+1}}\right)=\left(\frac{b_{n+2}}{b_n}-1\right)\left(1-\frac{b_{n-1}}{b_{n+1}}\right)$.

8. Reduce $\frac{351}{965}$ to the form of a continued fraction, and find the series of convergents.

9. Converge to $\frac{2723}{1799}$.

10. If in a continued fraction the quotients $q_2, q_3, \ldots q_{r+1}$, corresponding to the convergents $\frac{N_2}{D_2}, \frac{N_3}{D_3}$, etc., be all equal, then

$$\frac{N_{r+1}N_r-N_1N_0}{D_{r+1}D_r-D_1D_0}=\frac{N_1^2+N_2^2+\ldots+N_r^2}{D_1^2+D_2^2+\ldots+D_r^2}.$$

11. Form the convergents to $\frac{13164}{96507}$.

13. If $1, \frac{p_1}{q_1}, \frac{p_2}{q_2}$, etc. be the convergents of $\sqrt{3}$, prove that,

(1) $p_{2n+1}-p_1=p_{2n}+p_{2(n-1)}+p_{2(n-2)}+\ldots+p_2$,

(2) $p_{2n-1}-1=2\{p_{2n-1}+p_{2n-3}+p_{2n-5}+\ldots+p_1\}$,

(3) $p_{2n+1}-p_1=n+2\{p_{2n-1}+2p_{2n-3}+3p_{2n-5}+\ldots$
$+(n+1)p_2+np_1\}$

146. Prop. *The convergents of an odd order are less, and those of an even order greater, than the continued fraction; but each convergent differs from it less than any preceding one.*

Let x denote the continued fraction;
y_n, and a_n, the complete, and partial, quotients of the nth order;
$\frac{p_n}{q_n}, \frac{p_{n-1}}{q_{n-1}}, \frac{p_{n-2}}{q_{n-2}}$ the convergents of the nth, $(n-1)$th, $(n-2)$th orders.

Then x differs from $\frac{p_n}{q_n}$ only in having y_n instead of a_n;

$$\therefore\ x=\frac{y_n p_{n-1}+p_{n-2}}{y_n q_{n-1}+q_{n-2}};$$

$$\therefore\ y_n(xq_{n-1}-p_{n-1})=(p_{n-2}-xq_{n-2});$$

$$\therefore\ \frac{q_{n-1}}{q_{n-2}}y_n=\frac{\frac{p_{n-2}}{q_{n-2}}-x}{x-\frac{p_{n-1}}{q_{n-1}}}.$$

Now, first, q_{n-1}, q_{n-2}, y_n are all positive; $\therefore$ $\frac{q_{n-1}}{q_{n-2}}y_n$ is positive;
$\therefore \frac{p_{n-2}}{q_{n-2}}-x$ and $x-\frac{p_{n-1}}{q_{n-1}}$ are both positive, or both negative;
$\therefore$ if one convergent is greater, the next is less, than x, and *vice versâ*, but the first convergent is less than x, Art. 142, *Obs.* 3; $\therefore$ the second is greater, the third less, and so on; *i.e.*, all convergents of an odd order are less than x,
and „ „ even „ greater „ .

Secondly, y_n is >1, and, Art. 145, Cor., $\frac{q_{n-1}}{q_{n-2}}$ is also >1;

$\therefore\ \frac{q_{n-1}}{q_{n-2}}y_n$ is >1;

$\therefore\ \frac{p_{n-2}}{q_{n-2}}-x$ is numerically greater than $x-\frac{p_{n-1}}{q_{n-1}}$;

$\therefore\ \frac{p_{n-1}}{q_{n-1}}$ is nearer in value to x than $\frac{p_{n-2}}{q_{n-2}}$ is.

Hence the convergents continually approach, *or converge to*, the

value of the continued fraction. It is from this fact that they derive their name of convergents, or converging fractions.

COR. All convergents after the second are greater than a_1 and less than $a_1+\frac{1}{a_2}$.

147. Hence by taking the successive convergents we obtain nearer and nearer approximations to the value of the continued fraction. It may then be asked what error do we make in taking any particular convergent instead of the whole fraction? Before we answer this question we must prove the following propositions.

148. PROP. *If* p_n *and* q_n *be the numerator and denominator of the* n*th convergent, calculated according to* Art. 145, *then*

$$p_nq_{n-1}-q_np_{n-1}=(-1)^n.$$

We have $p_2q_1-q_2p_1=a_1a_2+1-a_1a_2=1$, and $\therefore$ the law holds when $n=2$.

Suppose it holds when $n=r$, *i.e.*, $p_rq_{r-1}-q_rp_{r-1}=(-1)^r$.

Now $p_{r+1}=a_{r+1}p_r+p_{r-1}$, $q_{r+1}=a_{r+1}q_r+q_{r-1}$;

$$\begin{aligned}\therefore p_{r+1}q_r-q_{r+1}p_r &= (a_{r+1}p_r+p_{r-1})q_r-(a_{r+1}q_r+q_{r-1})p_r,\\ &= p_{r-1}q_r-q_{r-1}p_r,\\ &= (-1)(p_rq_{r-1}-q_rp_{r-1})=(-1)^{r+1};\end{aligned}$$

$\therefore$ if the law holds when $n=r$, it holds when $n=r+1$; but it *does* hold when $n=2$; $\therefore$ also when $n=3$; $\therefore$ when $n=4$, and so on generally.

COR. 1. The rule in Art. 145 for finding the values of the convergents gives them in their lowest terms.

For if p_n and q_n had a common factor other than 1, it would exactly divide $p_nq_{n-1}-q_np_{n-1}$, *i.e.* $(-1)^n$, which is impossible.

COR. 2. The difference between two consecutive convergents is a fraction having unity for its numerator, and for its denominator the product of the denominators of the two convergents.

For $\frac{p_n}{q_n}-\frac{p_{n-1}}{q_{n-1}}=\frac{p_nq_{n-1}-p_{n-1}q_n}{q_nq_{n-1}}=(-1)^n\frac{1}{q_nq_{n-1}}$.

COR. 3. Now the continued fraction lies in value between $\frac{p_n}{q_n}$ and $\frac{p_{n-1}}{q_{n-1}}$; $\therefore$ it differs from $\frac{p_{n-1}}{q_{n-1}}$ by a number less than $\frac{1}{q_n q_{n-1}}$, and $\therefore$, *a fortiori*, since q_n is $> q_{n-1}$, by a number less than $\frac{1}{q^2_{n-1}}$.

Hence the error we make in taking $\frac{p_{n-1}}{q_{n-1}}$ instead of the whole fraction is less than $\frac{1}{q^2_{n-1}}$, and less even than $\frac{1}{q_n q_{n-1}}$.

149. PROP. *Any convergent is a nearer approximation to the value of the continued fraction than any other fraction whose denominator is less than that of the convergent.*

Let $\frac{p_n}{q_n}$, $\frac{p_{n-1}}{q_{n-1}}$ be two consecutive convergents, $\frac{a}{b}$ any other fraction such that $b < q_n$, a and b being positive integers.

If $\frac{a}{b}$ is nearer to the continued fraction than $\frac{p_n}{q_n}$, it is also nearer than $\frac{p_{n-1}}{q_{n-1}}$ (Art. 146); $\therefore$ it lies between $\frac{p_n}{q_n}$ and $\frac{p_{n-1}}{q_{n-1}}$;

$$\therefore \text{ numerically, } \frac{a}{b} - \frac{p_{n-1}}{q_{n-1}} < \frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}};$$

$$\therefore aq_{n-1} - bp_{n-1} < \frac{b}{q_n}(p_n q_{n-1} - q_n p_{n-1})$$

$$< \frac{b}{q_n} \text{ (Art. 148).}$$

Now a, b, p_{n-1}, q_{n-1} are all integers; $\therefore aq_{n-1} - bp_{n-1}$ is an integer. And $b < q_n$; $\therefore \frac{b}{q_n}$ is a proper fraction. Hence we have an integer numerically less than a proper fraction, which is absurd;

$$\therefore \frac{p_n}{q_n} \text{ is nearer than } \frac{a}{b} \text{ to the continued fraction.}$$

150. Thus by converting any given number into a continued fraction, and proceeding to the proper convergent, we obtain a fraction differing from the given number by less than an assigned difference, and by so doing we have found a fraction nearer in value to the given number than any other fraction having so small a denominator. This is a great advantage. For example, consider the number 3·14159, converting it into a continued fraction we have the following quotients: 3, 7, 15, 1, etc.;

$$\therefore \text{ the convergents are } 3, \frac{22}{7}, \frac{333}{106}, \frac{355}{113}, \text{ etc.};$$

$\therefore$ the fraction $\frac{355}{113}$ differs from the true value of 3·14159 by less than $\frac{1}{(113)^2}$, *i.e.* $\frac{1}{12769}$; and no fraction can be found with so small a denominator which does not differ from 3·14159 by a larger number.

EXERCISES.—XXVIII.

1. Taking 2·7182818 for the exact value of e, show that $\frac{107}{39}$ differs from it by less than $\frac{1}{2769}$; and $\frac{2763}{1001}$ by less than one millionth.

2. Find a fraction differing from $\sqrt{2}$ by less than $\frac{1}{500}$.

3. Show that $\frac{268}{65}$ is an approximation within $\frac{1}{34300}$ to the value of $\sqrt{17}$.

4. What error do we make in taking $\frac{161}{72}$ as the value of $\sqrt{5}$?

5. Express $\sqrt{6}$ as a continued fraction; and find the fraction nearest to it which has not more than 3 figures in its numerator.

6. Mars and the Earth revolve round the sun in 686·980 days and 365·256 days respectively. Find the fraction measuring the ratio of these periods most nearly, which has the largest denominator containing 2 digits; and show that the error in taking this measure is $<\frac{1}{22436}$.

7. Assuming that the Earth and Venus revolve round the sun in 365.25 and 88 days respectively, show that while Venus makes 83 revolutions the Earth makes 20 nearly, and that still more nearly 303 and 73 revolutions will be made simultaneously.

8. Two smiths begin to strike their anvils together. The one gives 12 strokes in 7′, the other 17 strokes in 9′. What strokes of each most nearly coincide in the first half-hour?

9. Two scales, whose zero points coincide, are placed side by side, and the space between consecutive divisions in one is to that in the other as 1 : 1.1543. Apply the principle of converging fractions to find those which most nearly coincide.

10. Find a series of fractions converging to the ratio of 5 hrs. 48′. 51″ to 24 hours.

11. A metre $=3·2809$ feet. Show that a kilometre is greater than $\frac{3}{5}$, and less than $\frac{5}{8}$, of a mile.

12. Express, within an error of $\frac{1}{184}$, the value of the ratio of £3. 7s. 5d. to 17s. 5d., by a fraction having one digit in its denominator.

13. A clock, which originally beats seconds, will under certain circumstances lose $\frac{1200}{667}$ beats in an hour. Show that this is about 9 beats in 5 hours.

14. Find $\sqrt{11}$ correct to two places of decimals.

15. Find x correct to four places of decimals, in the equation $5x^2=3$.

151. The following method gives the superior limit for the error (Art. 148, Cor. 3), and at the same time an inferior limit for it.

If y_n be the nth complete quotient to the continued fraction (x),

$$x-\frac{p_{n-1}}{q_{n-1}}=\frac{p_n y_n+p_{n-1}}{q_n y_n+q_{n-1}}-\frac{p_{n-1}}{q_{n-1}}$$

$$=\frac{y_n(p_n q_{n-1}-p_{n-1}q_n)}{q_{n-1}(q_n y_n+q_{n-1})}$$

$$=\frac{(-1)^n y_n}{q_{n-1}(q_n y_n+q_{n-1})} \qquad \text{(Art. 148)},$$

$$=\frac{(-1)^n}{q_{n-1}\left(q_n+\frac{1}{y_n}q_{n-1}\right)}.$$

Now y_n is finite and >1; $\therefore \frac{1}{y_n}$ is >0 and <1;

$\therefore$ the error *numerically* $<\frac{1}{q_{n-1}q_n}$ and $>\frac{1}{q_{n-1}(q_n+q_{n-1})}$.

152. Prop. *If* $\frac{p}{q}$, $\frac{p'}{q'}$ *be two consecutive convergents to a continued fraction* x, *then* $pp'-qq'x^2$ *is positive, or negative, according as* $\frac{p}{q}$ *is greater, or less, than* $\frac{p'}{q'}$.

Let y be the complete quotient of the order next after $\frac{p'}{q'}$, then

$$x=\frac{p'y+p}{q'y+q};$$

$$\therefore \frac{pp'}{qq'}-x^2=\frac{1}{qq'(q'y+q)^2}\{pp'(q'y+q)^2-qq'(p'y+p)^2\}$$

$$=\frac{1}{qq'(q'y+q)^2}\{y^2p'q'(pq'-qp')+pq(qp'-pq')\}$$

$$=\frac{1}{(q'y+q)^2}(y^2p'q'-pq)\left(\frac{p}{q}-\frac{p'}{q'}\right).$$

But $y>1$, and $p'>p$, $q'>q$; $\therefore$ $y^2p'q'-pq$ is positive; $\therefore$ $pp'-qq'x^2$ is positive, or negative, according as $\frac{p}{q}$ is greater, or less, than $\frac{p'}{q'}$.

153. *Ex.* To investigate the law of formation of the convergents of

$$\frac{a_1}{b_1+}\frac{a_2}{b_2+}\frac{a_3}{b_3+}\text{ etc.}$$

We have
$$\frac{p_1}{q_1}=\frac{a_1}{b_1},$$
$$\frac{p_2}{q_2}=\frac{a_1b_2}{b_1b_2+a_2},$$
$$\frac{p_3}{q_3}=\frac{b_3(a_1b_2)+a_3a_1}{b_3(b_1b_2+a_2)+a_3b_1}.$$

This last obeys the law

$$p_n=b_np_{n-1}+a_np_{n-2},\quad q_n=b_nq_{n-1}+a_nq_{n-2}.$$

Suppose this law to hold for the nth convergent, from which the $(n+1)$th differs only in having $b_n+\frac{a_{n+1}}{b_{n+1}}$ instead of b_n;

$$\therefore \frac{p_{n+1}}{q_{n+1}}=\frac{b_{n+1}a_np_{n-2}+(b_nb_{n+1}+a_{n+1})p_{n-1}}{b_{n+1}a_nq_{n-2}+(b_nb_{n+1}+a_{n+1})q_{n-1}}$$
$$=\frac{a_{n+1}p_{n-1}+b_{n+1}p_n}{a_{n+1}q_{n-1}+b_{n+1}q_n};$$

$\therefore$, if the nth convergent obeys this law, the $(n+1)$th does also; but the 3rd *does* obey the law; $\therefore$ all succeeding convergents do also.

For the continued fraction $\frac{a_1}{b_1-}\frac{a_2}{b_2-}\frac{a_3}{b_3-}$ etc., we should take $p_n=b_np_{n-1}-a_np_{n-2}$.

EXAMPLES.—XXIX.

1. If $\frac{p_n}{q_n}$ be the nth convergent to $\frac{a_1}{b_1+}\frac{a_2}{b_2+}\frac{a_3}{b_3+}$ etc., obtained by the rule in Art. 153, then $p_n q_{n+1} - q_n p_{n+1} = (-1)^n a_1 a_2 \ldots a_{n+1}$. Hence it follows that $\frac{p_n}{q_n}$ and $\frac{p_{n+1}}{q_{n+1}}$ need not be in their lowest terms.

2. If $\frac{\beta_1}{\alpha_1+}\frac{\beta_2}{\alpha_2+}$ etc. be such a fraction that in all cases $\beta_{n+1} = 1 + \alpha_n$, and $\frac{p_n}{q_n}$ be the nth convergent to it, prove that $p_n + \beta_1 q_n = \beta_1 \beta_2 \ldots \beta_{n+1}$.

3. If $\frac{p_n}{q_n}$ denote the nth convergent to a continued fraction (x) of the form $a_1 + \frac{1}{a_2+}\frac{1}{a_3+}$ etc., y_n the corresponding complete quotient, then $y_2 y_4 \ldots y_n = (-1)^n \frac{xq_1 - p_1}{xq_{n-1} - p_{n-1}}$.

4. If $\frac{N_{n-2}}{D_{n-2}}, \frac{N_{n-1}}{D_{n-1}}, \frac{N_n}{D_n}, \frac{N_{n+1}}{D_{n+1}}$ are four consecutive convergents to a continued fraction, and $q_{n-2}, q_{n-1}, q_n, q_{n+1}$ the corresponding quotients, such that $q_{n+1} = q_n$, prove that

$$D^2_n + D^2_{n-1} = D_n D_{n-2} + D_{n+1} D_{n-1}.$$

XV

Recurring Continued Fractions.

154. We have before (Art. 143) shown how to convert a quadratic surd into a continued fraction. We proceed to discuss the properties of the fraction produced by this process.

Let N be a positive integral, but not a square, number. Then

$$\sqrt{N}=a+\sqrt{N}-a=a+\frac{N-a^2}{\sqrt{N}+a}=a+\frac{r_1}{\sqrt{N}+a_1},$$

where a is the greatest integer in $\sqrt{N}$, and $a_1=a$, $r_1=N-a^2$.

Again $\dfrac{\sqrt{N}+a_1}{r_1}=b_1+\dfrac{\sqrt{N}-a_2}{r_1}=b_1+\dfrac{r_2}{\sqrt{N}+a_2}$,

where b_1 is the greatest integer in $\dfrac{\sqrt{N}+a_1}{r_1}$,

$$\text{and } a_2=r_1b_1-a_1, \quad r_2=\frac{N-a_2^2}{r_1}.$$

If this process be continued, the nth line will be

$$\frac{\sqrt{N}+a_{n-1}}{r_{n-1}}=b_{n-1}+\frac{\sqrt{N}-a_n}{r_{n-1}}=b_{n-1}+\frac{r_n}{\sqrt{N}+a_n},$$

where b_{n-1} is the greatest integer in $\dfrac{\sqrt{N}+a_{n-1}}{r_{n-1}}$,

$$\text{and } a_n=r_{n-1}b_{n-1}-a_{n-1}, \quad r_n=\frac{N-a_n^2}{r_{n-1}}.$$

Thus $\sqrt{N}=a+\dfrac{1}{b_1+}\,\dfrac{1}{b_2+}\cdots\dfrac{1}{b_{n-1}+}\,\dfrac{1}{b_n+}\cdots$

Cor. 1. The symbols a, b_1, b_2, . . . b_{n-1}, . . . and r_1 all represent positive integers.

Cor. 2. It is evident that, if a complete quotient is found to be identical in its terms with any preceding one, all quotients

succeeding this will be continually repeated in the same order, and we shall have what is called a *recurring*, or *periodic*, continued fraction.

Cor. 3. If $r_{n-1}=1$, then $a_n=a$, $r_n=r_1$.

For then $a_n+a_{n-1}=b_{n-1}$
$$=\text{the greatest integer in } \sqrt{N}+a_{n-1}$$
$$=a+a_{n-1};$$
$\therefore\ a_n=a$, and $r_n=N-a^2=r_1$.

Hence the $(n+1)$th complete quotient would be identical with the second, and we should have a recurrence.

155. Prop. *If* $\frac{\mathrm{p}}{\mathrm{q}}$, $\frac{\mathrm{p}'}{\mathrm{q}'}$ *be any two consecutive convergents to* $\sqrt{\mathrm{N}}$, *and* $\frac{\sqrt{\mathrm{N}}+\mathrm{a}''}{\mathrm{r}''}$ *the complete quotient of the order next after* $\frac{\mathrm{p}'}{\mathrm{q}'}$, *then* $\mathrm{a}''=\pm(\mathrm{pp}'-\mathrm{qq}'\mathrm{N})$, *and* $\mathrm{r}''=\pm(\mathrm{q}'^2\mathrm{N}-\mathrm{p}'^2)$;
the upper, or lower, sign being taken according as $\frac{\mathrm{p}}{\mathrm{q}}$ *is* $>$, *or* $<$, $\frac{\mathrm{p}'}{\mathrm{q}'}$.

For, as in Art. 146 $$\sqrt{N}=\frac{p'\dfrac{\sqrt{N}+a''}{r''}+p}{q'\dfrac{\sqrt{N}+a''}{r''}+q}=\frac{p'\sqrt{N}+p'a''+pr''}{q'\sqrt{N}+q'a''+qr''};$$
$$\therefore\ q'N+(q'a''+qr'')\sqrt{N}=p'\sqrt{N}+p'a''+pr''.$$

Equating the rational, and the irrational, parts on the two sides of this equation, we have
$$q'a''+qr''=p', \qquad p'a''+pr''=q'N;$$
$$\therefore\ a''(pq'-qp')=pp'-qq'N, \qquad r''(pq'-qp')=q'^2N-p'^2;$$
$$\text{but } pq'-qp'=\pm 1 \qquad \text{(Art. 148)};$$
$$\therefore\ a''=\pm(pp'-qq'N). \qquad r''=\pm(q'^2N-p'^2),$$
according as $\frac{p}{q}$ is $>$, or $<$, $\frac{p'}{q'}$, *i.e.*, according as the order of $\frac{p'}{q'}$ is odd or even.

Cor. 1. Hence a'' and r'' are *positive* integers.

For $q'^2N-p'^2$ (since $\sqrt{N}$ lies between $\frac{p}{q}$ and $\left.\frac{p'}{q'}\right)$, and $pp'-qq'N$ (by Art. 152), are themselves positive, or negative, according as $\frac{p}{q}$ is $>$, or $<$, $\frac{p'}{q'}$; $\therefore a''$ and r'' are positive.

They are also integers, for p, p', q, q', N are all integers.

Combining this with Cor. 1 of Art. 154, we see that each of the symbols in the three series $a, b_1, b_2, \ldots, a_1, a_2, a_3, \ldots, r_1, r_2, \ldots$, are positive integers.

Cor. 2. The number of values of a'' cannot exceed a.

For $a''^2=N-r'r''$. But $r'r''$ is positive; $\therefore a''^2$ is $<N$, and a is the greatest integer in $\sqrt{N}$;

$$\therefore a'' \text{ cannot be greater than } a;$$

i.e. it cannot have values other than 1, 2, 3 . . . a, which are a in number.

Cor. 3. The number of values of r'' cannot exceed $2a$.

For $r'=\dfrac{a''+a'}{b'}$. But from Cor. 2 $a''+a'$ cannot be $>2a$,

and (Cor. 1 of Art. 154) b' cannot be <1;

$$\therefore r' \text{ cannot be greater than } 2a;$$

i.e. it cannot have values other than 1, 2 . . . $2a$ which are $2a$ in number.

Cor. 4. The number of complete quotients cannot exceed $2a^2$.

For, Cor. 2, $\sqrt{N}+a''$ cannot have more than a different values,
and, Cor. 3 $\quad r'' \quad$,, $\quad$,, $\quad 2a \quad$,, $\quad$;
$\therefore$ combining all the different numerators with all the different denominators, we cannot form more than $2a^2$ different complete quotients.

Hence after $2a^2$ complete quotients, at most, we must have one that has occurred before.

Hence every quadratic surd gives rise to a recurring continued fraction.

156. Prop. *The periodic part begins with the second quotient, and ends with one which is double the first.*

Since a recurrence must take place, suppose the $(n+1)$th complete quotient is a repetition of the $(m+1)$th, so that

$$a_m = a_n, \qquad r_m = r_n, \quad . \quad . \quad . \quad (1).$$

Then the nth complete quotient must be a repetition of the mth.

$$\text{For } r_{m-1} = \frac{N - a_m^2}{r_m} = \frac{N - a_n^2}{r_n} = r_{n-1}, \quad . \quad . \quad . \quad (2).$$

Also $a_m + a_{m-1} = r_{m-1} b_{m-1}$,

$a_n + a_{n-1} = r_{n-1} b_{n-1}$;

$$\therefore \frac{a_{m-1} - a_{n-1}}{r_{n-1}} = b_{m-1} - b_{n-1} = \text{zero, or an integer}, \quad . \quad (3).$$

Again if $\frac{p}{q}$, $\frac{p'}{q'}$ denote the $(n-2)$th and $(n-1)$th convergents, then $q'a_{n-1} + qr_{n-1} = p'$, Art. 155, if n be greater than 2;

$$\therefore \frac{q}{q'} r_{n-1} = \frac{p'}{q'} - a_{n-1}.$$

But $\frac{p'}{q'}$ is $> a$; $\therefore \frac{q}{q'} r_{n-1} > a - a_{n-1}$,

and $q < q'$; $\therefore r_{n-1} > a - a_{n-1}$;

$$\therefore, \text{ if } n - 1 > 1, \ \frac{a - a_{n-1}}{r_{n-1}} < 1, \quad . \quad . \quad . \quad . \quad (4).$$

Similarly, if $m > 2$, $\frac{a - a_{m-1}}{r_{m-1}} < 1$.

Now $r_{n-1} = r_{m-1}$;

$$\therefore \frac{a - a_{m-1}}{r_{n-1}} < 1, \quad . \quad . \quad . \quad . \quad (5).$$

Hence the difference of (4) and (5) is numerically less than 1;

$\therefore \frac{a_{m-1} - a_{n-1}}{r_{n-1}} =$ zero, or a proper fraction, and therefore, by (3), must be zero;

$$\therefore a_{m-1} = a_{n-1}.$$

Hence, if $m > 2$, the mth complete quotient recurs at the nth.

In the same manner we can show that the $(m-1)$th recurs at the $(n-1)$th, and thus, going back, that each complete quotient recurs $n-m$ terms further on, until we come to the third.

To show that the second is repeated we proceed as follows.

We have $a_2 = a_{2+n-m}$, $r_2 = r_{2+n-m}$;

$\therefore$, as in (2), $r_1 = r_{1+n-m}$,

and, as in (3), $\dfrac{a_1 - a_{1+n-m}}{r_1} =$ zero, or an integer,

and, as in (4), $\dfrac{a - a_{1+n-m}}{r_{1+n-m}} < 1$.

Now $a = a_1$, and $r_1 = r_{1+n-m}$;

$$\therefore \frac{a_1 - a_{1+n-m}}{r_1} < 1\,; \quad \therefore \frac{a_1 - a_{1+n-m}}{r_1} = \text{zero}\,;$$

$$\therefore a_1 = a_{1+n-m}.$$

Thus the second quotient, as well as all that follow it, is repeated in the same order.

But the first quotient is not repeated before the second.

For let $\dfrac{\sqrt{N} + a_s}{r_s}$ be the complete quotient which immediately precedes the recurrence of the second, so that

$\dfrac{\sqrt{N} + a_s}{r_s}$ and $\dfrac{\sqrt{N} + a_1}{r_1}$ are consecutive complete quotients;

$$\therefore r_s = \frac{N - a^2}{r_1} = 1\,;$$

$$\therefore, \text{ by } (4),\ a - a_s < 1\,;$$

but a_s cannot be greater than a by Art. 155, Cor. 2, and must be integral by Cor. 1;

$\therefore a - a_s$ cannot be fractional, or negative, and $\therefore = 0$;

$$\therefore a_s = a;\ \text{see also Art. 154, Cor. 2};$$

$\therefore$ the greatest integer in $\dfrac{\sqrt{N} + a_s}{r_s}$ is $2a$ instead of a as it would be if the recurrence began with the first quotient.

Hence the periodic part begins with the second quotient, and ends with a quotient which is double the first.

COR. If $\frac{p}{q}$ be the convergent preceding that corresponding to the quotient a_s at the end of the period, then since $r_s=1$, we have (Art. 155) $q^2N-p^2=\pm 1$, according as $\frac{p}{q}$ is of an odd, or an even, order.

We may call $\frac{p}{q}$ the *penultimate* convergent of the recurring period.

157. PROP. *Every periodic continued fraction is a root of a quadratic equation of which the coefficients are rational.*

Let x denote the fraction, y the periodic part,
$a, b, c, \ldots m, n$, the quotients of the non-periodic part,
$\alpha, \beta, \gamma, \ldots \mu, \nu$, „ „ periodic part,

$$\text{so that } x=a+\frac{1}{b+}\frac{1}{c+}\cdots\frac{1}{m+}\frac{1}{n+}\frac{1}{y},$$

$$y=\alpha+\frac{1}{\beta+}\frac{1}{\gamma+}\cdots\frac{1}{\mu+}\frac{1}{\nu+}\frac{1}{y}.$$

Let $\frac{P}{Q}$, $\frac{R}{S}$ be the convergents to x corresponding to m and n.

Hence y is the complete quotient of the next order to $\frac{R}{S}$;

$$\therefore x=\frac{Ry+P}{Sy+Q} \text{ (as in Art. 146)}, \qquad (1).$$

Let $\frac{p}{q}$, $\frac{r}{s}$ be the convergents to y corresponding to μ and ν;

$$\therefore \text{ in a similar way } y=\frac{ry+p}{sy+q}, \qquad (2);$$

$\therefore$, eliminating y between (1) and (2), we obtain a quadratic for x.

158. Hence every recurring continued fraction is equivalent to an expression $\frac{\pm L+\sqrt{N}}{M}$ where L, M, N are positive integers, and N is not a square number.

159. For the general proof of the converse proposition, *that every expression of the above form can be converted into a recurring continued fraction*, the student is referred to Serret's *Cours d'Algèbre Supérieure.*

We may remark that we have proved it, (Art. **155**, Cor. 4), in the case in which $L=0$ and $M=1$, and when L and M have such values as would make the above expression one of the complete quotients obtained during the conversion of $\sqrt{N}$ into a continued fraction.

EXAMPLES.—XXX.

1. Find the value of $1+\frac{1}{2+}\frac{1}{3+}\frac{1}{4+}\frac{1}{1+}\frac{1}{2+}$ etc., and find the first six convergents.

2. Prove that $2a+\frac{1}{a+}\frac{1}{4a+}\frac{1}{a+}$ etc., $=2\sqrt{1+a^2}$; and that the second convergent differs from the true value by a number less than $\frac{1}{a(4a^2+1)}$; and thence, by making $a=7$, show that $\frac{99}{70}$ differs from $\sqrt{2}$ by a number $<\frac{1}{13790}$.

3. If $a, \frac{p_1}{q_1}, \frac{p_2}{q_2}, \ldots \frac{p_n}{q_n}, \ldots$ be successive convergents to $\sqrt{a^2+1}$, prove that $\frac{p_{n+1}-p_{n-1}}{p_n}=2a.$

4. Prove that $\{p(p+4)\}^{\frac{1}{2}}=p+\frac{2}{1+}\frac{1}{p+}\frac{1}{1+}\cdots$

5. If $\frac{p_n}{q_n}$ be the nth convergent to the infinite continued fraction

$$\frac{1}{a+}\frac{1}{b+}\frac{1}{a+} \text{ etc.,}$$

show that $p_{2n+2}-2p_{2n}+p_{2n-2}=abp_{2n}$.

6. Show that $a+\frac{1}{1+}\frac{1}{b+}\frac{1}{a+}\text{etc.} : b+\frac{1}{1+}\frac{1}{a+}\frac{1}{b+}\text{etc.}$
$=a+1:b+1.$

7. If $\frac{p_r}{q_r}$ denote the rth convergent to $\frac{\sqrt{5}+1}{2}$, show that

$$p_2+p_5+\ \ldots\ +p_{2n-1}=p_{2n}-p_2$$
$$q_2+q_5+\ \ldots\ +q_{2n-1}=q_{2n}-q_2.$$

8. Show that, if n be any number,

$$\left(n+\frac{1}{2n+}\frac{1}{2n+}\frac{1}{2n+}\text{etc.}\right)^2-\left(n-\frac{1}{2n-}\frac{1}{2n-}\frac{1}{2n-}\text{etc.}\right)^2=2.$$

9. Show that $\frac{3a}{2}+\frac{1}{a+}\frac{1}{3a+}\frac{1}{a+}\frac{1}{3a+}\text{etc.}=\frac{1}{2}\sqrt{9a^2+12}.$

10. Prove that the value of the fraction

$$\frac{\frac{1}{x+}\frac{1}{4x+}\frac{1}{x+}\frac{1}{4x+}\ldots}{\frac{1}{2x+}\frac{1}{2x+}\ldots}$$

is independent of the value of x.

11. Prove that $7+\frac{1}{14+}\frac{1}{14+}\text{etc.}=5\left\{1+\frac{1}{2+}\frac{1}{2+}\text{etc.}\right\}.$

12. Find the 6th convergent to the positive root of $2x^2-3x-6=0$.

13. If $x=\frac{1}{a_1+}\frac{1}{a_2+}\frac{1}{a_1+}\frac{1}{a_2+}\ldots,$

$$y=\frac{1}{2a_1+}\frac{1}{2a_2+}\frac{1}{2a_1+}\frac{1}{2a_2+}\ldots,$$

$$z=\frac{1}{3a_1+}\frac{1}{3a_2+}\frac{1}{3a_1+}\frac{1}{3a_2+}\ldots,$$

prove that $x(y^2-z^2)+2y(z^2-x^2)+3z(x^2-y^2)=0.$

160. *Ex.* Show that the nth convergent to the infinite continued fraction

$$\frac{1}{a-}\,\frac{1}{a-}\,\frac{1}{a-}\text{ etc.}$$

is $\dfrac{\alpha^n-\beta^n}{\alpha^{n+1}-\beta^{n+1}}$, where $\alpha+\beta=a$, $\alpha\beta=1$.

Putting $n=1$ and 2, we see that this formula gives the first and second convergents.

Suppose then it gives the rth and the $(r-1)$th convergents.

$$\text{Then } \frac{p_{r+1}}{q_{r+1}}=\frac{a(\alpha^r-\beta^r)-(\alpha^{r-1}-\beta^{r-1})}{a(\alpha^{r+1}-\beta^{r+1})-(\alpha^r-\beta^r)}$$

$$=\frac{(\alpha+\beta)(\alpha^r-\beta^r)-\alpha^{r-1}+\beta^{r-1}}{(\alpha+\beta)(\alpha^{r+1}-\beta^{r+1})-\alpha^r+\beta^r}$$

$$=\frac{\alpha^{r+1}-\alpha\beta\beta^{r-1}+\beta\alpha\alpha^{r-1}-\beta^{r+1}-\alpha^{r-1}+\beta^{r-1}}{\alpha^{r+2}-\alpha\beta\beta^r+\alpha\beta\alpha^r-\beta^{r+2}-\alpha^r+\beta^r}$$

$$=\frac{\alpha^{r+1}-\beta^{r+1}}{\alpha^{r+2}-\beta^{r+2}},\text{ since } \alpha\beta=1.$$

Hence then in this case the law will also hold for the $(r+1)$th convergent; but it holds for the 1st two; $\therefore$, by induction, it holds for all succeeding convergents.

MISCELLANEOUS EXAMPLES.—XXXI.

1. If $\dfrac{3-\sqrt{5}}{2}$ be converted into a continued fraction, the first four convergents are $\dfrac{1}{2}$, $\dfrac{1}{3}$, $\dfrac{2}{5}$, $\dfrac{3}{8}$, and the nth is

$$4\frac{(\sqrt{5}+1)^n-(-\sqrt{5}+1)^n}{(\sqrt{5}+1)^{n+2}-(-\sqrt{5}+1)^{n+2}}.$$

2. The nth convergent to the fraction

$$\frac{r}{r+1-}\,\frac{r}{r+1-}\,\frac{r}{r+1-}\text{ etc.}$$

is $\dfrac{r^{n+1}-r}{r^{n+1}-1}$.

3. What is the limiting value of $x+\frac{1}{x+}\frac{1}{x+}$ etc., when x approaches zero?

4. If $u=a+\frac{b}{a+}\frac{b}{a+}\ldots$ to infinity and $v=a+\frac{1}{a+}\frac{1}{a+}\ldots$ to infinity, prove that $v=\frac{u^2-b}{u}+\cfrac{1}{\cfrac{u^2-b}{u}+\ldots}$

5. Show that the nth convergent to the continued fraction

$$\frac{1}{2+}\frac{1}{2+}\frac{1}{2-}\text{ etc. is }\frac{(1+\sqrt{2})^n-(1-\sqrt{2})^n}{(1+\sqrt{2})^{n+1}-(1-\sqrt{2})^{n+1}}.$$

6. Show that, if

$$x=\cfrac{y}{y^2+\cfrac{y}{y^2+\text{ etc., } ad\ inf.,}}\qquad y=\cfrac{x}{\cfrac{1}{x}-\cfrac{x}{\cfrac{1}{x}-\text{ etc., } ad\ inf.}}$$

7. If $a_1, a_2, \ldots a_n$ be n terms of a harmonical progression,

$$\frac{a_n}{a_{n-1}}=\frac{1}{2-}\frac{1}{2-}\text{ etc. }-\frac{a_2}{a_1}.$$

XVI

Indeterminate Equations of the First Degree.

161. In [Chapter xxii.] a method was given for finding positive integral solutions of *Indeterminate Equations* of the first degree involving two unknowns. We shall in the present and 17th chapters discuss this subject more fully, principally by means of the Properties of Continued Fractions proved in the two preceding chapters.

162. It is evident that an indeterminate equation admits of an infinite number of solutions, taking into account all the positive, negative, integral and fractional values which can be given to the unknown symbols involved so as to satisfy the equation.

The problems producing such equations, however, very often require us to take only the positive integral solutions.

163. For instance, consider this problem,—"If a bullock cost £10 and a sheep £2, find the number of animals I can buy for £50."

The *equation* produced is

$$10x+2y=50,$$
$$\text{or}\quad 5x+y=25.$$

Here evidently the *problem* allows us to take only the *positive integral* solutions of the *equation*, viz.—

$$x=0,\ y=25.$$
$$x=1,\ y=20.$$
$$x=2,\ y=15.$$
$$x=3,\ y=10.$$
$$x=4,\ y=\ 5.$$
$$x=5,\ y=\ 0.$$

We shall therefore give especial attention to such solutions.

164. In the instance given above, if I am allowed to buy only sheep, or only bullocks, we can take all six solutions; but if I must buy *some* bullocks and *some* sheep at the same time, we must reject the first and last solutions.

Such solutions as these two are called *zero* solutions, and will always be considered as included amongst positive integral solutions unless the contrary be indicated.

We shall discuss, in the present chapter, equations of the first order, and in the next, those of a higher order.

165. Every equation of the first degree involving two unknowns can be put into one or other of the four following forms,

$$ax+by=c\ (1);\qquad ax-by=c\ (2);$$
$$-ax+by=c\ (3);\quad -ax-by=c\ (4);$$

for we may always suppose c to be positive, since, if it be negative, we may change the signs on both sides of the equation, and thus reduce it to one of the above forms.

Again, we may suppose there is no factor common to a, b, and c, for if there is, as in Art. 163, we can divide each side by it, and thus reduce the equation to an equivalent one in which the coefficients have no common factor.

Further, if a and b have a common factor, none of the above forms are solvable in integers; for then, if x and y be integers, the left-hand sides are divisible by this common factor, but c is not, and this is an impossibility.

Of the above forms we see at once that (4) cannot have a *positive* solution, and (3) is essentially the same in form as (2); it will suffice then to consider the two first.

166. Prop. *If* $x=\alpha$, $y=\beta$ *be an integral solution of the equation* $ax+by=c$, *and* t *be an integer, then all integral solutions are included in the formulæ,* $x=\alpha-bt$, $y=\beta+at$, *by giving* t *all integral values.*

$$\text{For } ax+by=c=a\alpha+b\beta;$$
$$\therefore a(x-\alpha)=b(\beta-y);$$
$$\therefore \frac{x-\alpha}{y-\beta}=-\frac{b}{a};$$

but $\frac{b}{a}$ is a fraction in its lowest terms (Art. 165), hence, as in [Art. 163], $x-\alpha$ is some multiple of b, say bt, and $y-\beta$ is the same multiple, at, of a, *i.e.*,

$$x-\alpha=-bt, \quad y-\beta=at;$$
$$\therefore x=\alpha-bt, \quad y=\beta+at.$$

167. Similarly it can be shown that, if $x=\alpha$, $y=\beta$ be a solution of $ax-by=c$, all integral solutions are included in the formulæ,

$$x=\alpha+bt, \quad y=\beta+at.$$

168. If *positive* integral solutions only be required, we must give to t only such integral values as will make the above solutions positive integers.

169. If then we find any integral solution of a given equation, either by inspection, or by [Chap. XXII.], we may complete the solutions by Articles 166 and 167.

170. Cor. 1. We can see at once that $ax-by=c$ has an infinite number of positive integral solutions, for whatever finite values α and β may have, we can always find a value for t which will make $\alpha+bt$ and $\beta+at$ positive, and then all greater values of t will also make them positive.

Cor. 2. If $a+b>c$, it is impossible to find *positive* integral values of x and y which will make $ax+by$ as small as c; hence $ax+by=c$ cannot in this case be solved in *positive* integers.

171. Prop. *The number of positive integral solutions of* $ax+by=c$ *cannot exceed* $\frac{c}{ab}+1$.

For, if there is no positive integral solution, of course the theorem is true.

If there is one, let it be

$$x=\alpha, \quad y=\beta;$$
$$\therefore \ a\alpha+b\beta=c.$$

Then all other solutions may be obtained from the formulæ,

$$x=\alpha-bt, \quad y=\beta+at,$$

by giving t all integral values (including zero) between $\frac{\alpha}{b}$ and $-\frac{\beta}{a}$ (Art. 166 and 168).

Let m and n be the greatest integers in $\frac{\alpha}{b}$ and $\frac{\beta}{a}$.

Then t may have each of the following values,

$$m,\ m-1\ \ldots\ 1,\ 0,\ -1\ \ldots\ -(n-1),\ -n.$$

For each value of t we have one solution;

$$\therefore \text{ the number of solutions } =m+1+n.$$

$$\text{But } m \text{ is not } > \frac{\alpha}{b}, \text{ nor } n > \frac{\beta}{a};$$

$$\therefore \text{ the number of solutions is not } > \frac{\alpha}{b}+\frac{\beta}{a}+1;$$

$$\text{i.e. not } > \frac{a\alpha+b\beta}{ab}+1,$$

$$\text{not } > \frac{c}{ab}+1.$$

EXAMPLES.—XXXII.

1. In how many ways can a person who has only half-crowns and florins pay a debt of £2. 17s.?

2. In how many ways can £2. 15s. 6d. be paid by the person in the preceding question?

3. Show that the equation $49x+63y=491$ can have no solution in which x and y are both integers.

4. In how many ways can the sum of £3 be paid in half-crowns and shillings?

5. One solution of the equation $9x-13y=1$ is $x=3$, $y=2$. Find all the positive integral solutions.

6. Of all the values of x and y which form the positive integral solutions of the equation $ax-by=c$, the least value of x and the least value of y belong to the same pair.

[This is called the simplest solution.]

7. In how many different ways is it possible to pay £100 in half-guineas and sovereigns.

8. If $ax-by=c$ be solved in positive integers, show that the successive values of x are in A. P., of which b is the common difference, and *similarly* for the values of y.

9. How many crowns and half-crowns, whose diameters are respectively ·81 and ·666 of an inch, may be placed in a row close together so as to make a yard in length.

10. Determine the number of positive integral solutions of $2x+7y=100$.

11. Determine the positive integral solutions of $2x+7y=100$.

12. Find the greatest possible value of c when the equation $3x+5y=c$ has 8 solutions and no more.

13. Also when the equation $5x+7y=c$ has 10 and no more.

14. Determine the greatest number, which can be formed in 11 ways, by adding together a multiple of 13 and of 7.

15. A company of men and women pay altogether 1000 francs. The men pay 19 francs each, and the women 11. How many men and women were there in the company?

16. There are two unequal rods, one 5 feet long and the other 7. How many of each can be taken to make up a length of 123 feet?

17. Find the least number such that when divided by 11 there remains 3, and when divided by 17 there remains 10.

172. Prop. *To solve the equation* $\mathrm{ax-by=c}$ *in integers, by continued fractions.*

Convert the fraction $\frac{a}{b}$ into a continued fraction, and form the successive convergents to it, the last of which is itself.

Let $\frac{p}{q}$ be the penultimate convergent, then, according as its order is odd, or even, we have

$$aq-bp=\pm 1, \qquad . \qquad . \text{ (Art. 148)};$$

$$\therefore a(\pm cq)-b(\pm cp)=c,$$

the upper signs only, or the lower signs only, being taken;

$$\therefore x=\pm cq, \; y=\pm cp \text{ is one solution,}$$

and the general solution is

$$x=\pm cq+bt, \; y=\pm cp+at, \qquad . \text{ (Art. 167)},$$

where t is any integer.

Note.—If the upper signs are to be taken, $\frac{q}{b}$ is $>\frac{p}{a}$; and all negative values of t between 0 and $-\frac{cp}{a}$, and all positive values, will give *positive* solutions.

If the lower signs are to be taken, $\frac{q}{b}$ is $<\frac{p}{a}$, and all positive values of t not less than $\frac{cp}{a}$ will give *positive* solutions.

173. Prop. *To solve the equation* $\mathrm{ax+by=c}$ *in integers, by continued fractions.*

In the same way as in the preceding article, it can be shown that the general solution is

$$x=\pm cq-bt, \; y=\mp cp+at,$$

where t is any integer.

Note.—If the upper signs have to be taken, $\frac{q}{b}$ is $>\frac{p}{a}$, and all values of t between $\frac{cq}{b}$ and $\frac{cp}{a}$ will give *positive* solutions.

If the lower signs have to be taken, $\frac{q}{b}$ is $<\frac{p}{a}$, and all values of t between $-\frac{cq}{b}$ and $-\frac{cp}{a}$ will give *positive* solutions.

174. If either $a=1$ or $b=1$, this method fails, as we cannot then convert $\frac{a}{b}$ into a continued fraction; but we can at once solve the equation by inspection. For instance, consider the equation

$$ax+y=c;$$

$$\therefore y=c-ax;$$

$\therefore$ all integral values of x between 0 and $\frac{c}{a}$ will give positive integral values for y.

EXAMPLES.—XXXIII.

Apply the method of continued fractions for the solution of the following examples.

1. Find the positive integral solutions of the equations

(1) $8x+13y=159$. (2) $29x+17y=250$.

(3) $5x+3y=78$. (4) $6x+7y=122$.

(5) $24x+65y=243$. (6) $8x+65z=81$.

2. Find the simplest solutions of the equations

(1) $25x-16y=12$. (2) $39x-56y=11$.

(3) $17x-49y=-8$. (4) $49x-36y=11$.

3. Find the least possible solution in positive integers of the equation $355x-113y=3888$. What is the next smallest solution?

4. Find all the positive integral values of x and y, less than 50, which satisfy the equation

$$13x-11y=20.$$

5. Find the simplest solution of the equation

$$99x-100y=10.$$

6. In how many ways can a man, who has only 20 crown pieces, pay another, who has only 13 florins, the sum of 11s.? Which is the simplest solution?

175. *To solve the equation* $ax+by+cz=d$ *in positive integers.*

We have $ax+by=d-cz$.

Give z all positive integral values between 0 and $\frac{d}{c}$, and determine by the preceding Articles the corresponding positive integral values of x and y.

Ex. Solve in positive integers the equation $5x+20z+6y=187$.

We have $5x+20z=187-6y$. Hence y must be $<\frac{187}{6}$ or $31\frac{1}{6}$.

Again since 5 and 20 have a common factor, viz., 5, no value of y need be tried which does not make $187-6y$ a positive multiple of 5.

Now $5x+20z=187-6y=5(37-y)+2-y$;

$\therefore$ we can try $y=2, 7, 12, 17, 22, 27$;

$$\begin{aligned}\therefore\ x+4z&=37-\ 2 &&=35,\\ \text{or } &=37-\ 7-1&&=29,\\ \text{or } &=37-12-2&&=23,\\ \text{or } &=37-17-3&&=17,\\ \text{or } &= &&\ 11,\\ \text{or } &= &&\ \ 5.\end{aligned}$$

Each of these must now be solved in positive integers with regard to x and z.

EXAMPLES.—XXXIV.

Solve in positive integers the following five equations.

1. $15x+6y+20z=171$.
2. $3x+7y+17z=100$.
3. $2x+4y+5z=49$.
4. $\frac{x}{2}+\frac{y}{3}+\frac{z}{6}=12$.
5. $31x+11y+z=200$.

How many solutions are there of the following three equations?

6. $3x+5y+6z=64.$

7. $19x+14y+15z=100.$

8. $8x+4y+3z=49.$

9. Can the equation $6x+10y-15z=11$ be solved in positive integers?

176. *To solve the equations* $ax-by+cz=d$, $a'x+b'y+c'z=d'$ *in positive integers.*

Eliminate z; let the resulting equation be

$$Ax+By=D.$$

Let $x=a$, $y=\beta$ be a solution of it, then we can put (Art. **166, 167**),

$$x=a-Bt,\ y=\beta+At, \qquad . \quad . \qquad (1).$$

Substitute these expressions for x and y in either of the given equations, and thus obtain an equation involving t and z.

Obtain a positive integral solution of this equation, and thus by Art. **166, 167**, express t and z in terms of a new unknown (t'), and then by substituting for t, express x and y in terms of t'.

Now give to t' all the values which will make x, y, and z each equal to a positive integer.

EXAMPLES.—XXXV.

Solve the following pairs of simultaneous equations.

1. $\left.\begin{array}{r} 5x+4y+z=272, \\ 8x+9y+3z=656. \end{array}\right\}$

2. $\left.\begin{array}{r} 6x+7y+4z=22, \\ 11x+8y-6z=145. \end{array}\right\}$

3. $\left.\begin{array}{r} x-2y+z=5, \\ 2x+y-z=7. \end{array}\right\}$

4. In how many ways may the sum of £2 be paid in half-crowns, shillings, and sixpences, supposing 28 coins to be always used?

5. A farmer spends £100 in buying bullocks at £10 each, sheep at £1 each, and geese at 2s. 6d. each. He buys 100 head in all. How many of each does he buy?

MISCELLANEOUS EXAMPLES.—XXXVI.

1. In how many ways may the sum of £24. 15s. be paid in shillings and francs, supposing 26 francs to be equal to 21s.?

2. Find a number of three digits, which, when added together, make up 20.

3. Find a number of two digits, which if the digits be inverted and 9 added, shall be doubled.

4. In how many ways can a person pay a sum of £15 in half-crowns, shillings, and sixpences; so that the number of shillings and sixpences together shall equal the number of half-crowns?

5. Find a number which, when divided by 39, leaves 16 remainder, and when divided by 56 leaves a remainder 27.

6. Find a value for x which will make the expressions $\frac{3x-10}{7}$, $\frac{11x+8}{17}$, $\frac{16x-1}{5}$ all whole numbers.

7. Divide 100 into two parts, such that one part may be divisible by 7, and the other by 11.

8. A man buys a number of horses at £30 each, and a number of bullocks at £12, and finds that he has spent £8 more on bullocks than on horses. How many did he buy?

9. Find the number of ways in which I can mix 40 gallons of wine, some at 15s., some at 19s., and some at 12s. per gallon, so as to produce a mixture worth 16s. per gallon, an integral number of gallons of each sort being taken.

10. A certain sum of money consists of £x and y shillings,

and its nth part of £y and x shillings; find the values of n which will give sums properly answering the conditions of the problem.

11. In how many different ways may £11. 15s. be paid in guineas and crowns? If those values of x and y be taken whose sum is the least, show that the nth power of this sum may be expressed by the series

$$10^n\left\{1+n\frac{1}{3}+\frac{n(n+1)}{1.2}\frac{1}{3^2}+\frac{n(n+1)(n+2)}{1.2.3}\cdot\frac{1}{3^3}+\text{ etc. }\right\}.$$

12. Find by means of continued fractions the positive integral solutions of $23x-19y=97$.

13. Find three fractions whose denominators are 3, 4, and 5, and whose sum is $\frac{133}{60}$.

14. In how many different ways could a courtyard 20 yards long and 15 broad be paved with stones selected from two sets, the stones in one set being $3\frac{2}{3}$ feet long and 3 broad, and those in the other $4\frac{3}{4}$ feet long and 4 broad?

15. A table $7\frac{1}{2}$ feet long and $4\frac{2}{3}$ feet broad is to be covered with photographs selected from pictures measuring respectively $1\frac{1}{3}$ feet by 10 inches, $1\frac{2}{3}$ feet by 1 foot, and $2\frac{1}{3}$ feet by $3\frac{1}{3}$ feet. In how many ways may this be done?

XVII

Indeterminate Equations of the Second Degree.

177. In Art. 156, Cor. it was proved that if $\sqrt{N}$ be converted into a continued fraction, and $\frac{p}{q}$ denote the penultimate convergent of any recurring period, then $p^2-Nq^2=\pm 1$, according as $\frac{p}{q}$ is of an even, or odd, order. From this fact we are able to deduce a series of solutions of certain indeterminate equations of the second degree.

178. Prop. *To solve* $x^2-Ny^2=1$, *in positive integers, when* N *is a positive integer, but not a perfect square.*

Convert $\sqrt{N}$ into a continued fraction, and let $\frac{p}{q}$ be the penultimate convergent of the first recurring period.

Then $(p^2-Nq^2)^n=(\pm 1)^n=1$, n being *any* positive integer, or any positive *even* integer, according as the order of $\frac{p}{q}$ is even, or odd;

$$\therefore (p-\sqrt{N}q)^n(p+\sqrt{N}q)^n=(x-\sqrt{N}y)(x+\sqrt{N});$$

$\therefore$ any values of x and y which satisfy the equations

$$x-\sqrt{N}y=(p-\sqrt{N}q)^n,$$

$$\text{and } x+\sqrt{N}y=(p+\sqrt{N}q)^n,$$

$$\text{or } x=\frac{(p+\sqrt{N}q)^n+(p-\sqrt{N}q)^n}{2},$$

$$\text{and } y=\frac{(p+\sqrt{N}q)^n-(p-\sqrt{N}q)^n}{2\sqrt{N}},$$

will be solutions of the given equation.

179. *Ex.* To solve $x^2-13y^2=1$.

Here Art. 143, $\frac{p}{q}=\frac{18}{5}$, and is of the *fifth* order. Hence

$$x=\frac{(18+5\sqrt{13})^n+(18-5\sqrt{13})^n}{2},$$

$$y=\frac{(18+5\sqrt{13})^n-(18-5\sqrt{13})^n}{2\sqrt{13}},$$

where n must be zero, or an even integer.

Giving n the values 0 and 2, we get

$$x=1,\ 649,$$

$$y=0,\ 180;$$

and for every even integer which we might put for n we should obtain another pair of values for x and y.

180. Prop. *To solve* $\mathrm{x}^2-\mathrm{N}\mathrm{y}^2=-1$, *in positive integers, when* N *is positive integer, but not a perfect square.*

Convert $\sqrt{N}$ into a continued fraction, and let $\frac{p}{q}$ be the penultimate convergent of the first recurring period.

Then, if $\frac{p}{q}$ is of an odd order, $p^2-Nq^2=-1$, and $(p^2-Nq^2)^n=-1$, where n is any *odd* integer;

$$\therefore (p^2-Nq^2)^n=(x^2-Ny^2),$$

and the solutions will be of the same form as in Art. 178.

If, however, $\frac{p}{q}$ is of an even order, $(p^2-Nq^2)^n=1$ whatever integral value n may have, and the equation cannot be solved.

181. If A is the denominator of some one of the complete quotients which recur in converting $\sqrt{N}$ into a continued fraction, we can often obtain a solution for the equations $x^2-Ny^2=A$, . . . (1), $x^2-Ny^2=-A$, . . . (2).

Let $\frac{p'}{q'}$ be the convergent, in the first period, whose order is one less than that of the complete quotient in which A is the denominator.

Let $\frac{p''}{q''}$ be the similar convergent in the second period.

Then if the number of quotients in each period is odd, the orders of $\frac{p'}{q'}$, $\frac{p''}{q''}$ are, one odd, and the other even;

$$\therefore p'^2-Nq'^2=+A, \text{ and } p''^2-Nq''^2=-A,$$
$$\text{or } p'^2-Nq'^2=-A, \quad ,, \quad p''^2-Nq''^2=+A,$$

according as the order of $\frac{p'}{q'}$ is even, or odd, Art. 155.

In the first case $x=p'$, $y=q'$ is a solution of (1),
and $x=p''$, $y=q''$,, ,, (2).
In the second case $x=p''$, $y=q''$,, ,, (1),
and $x=p'$, $y=q'$,, ,, (2).

But if the number of quotients in each period is even, the orders of $\frac{p'}{q'}$, $\frac{p''}{q''}$ are, both odd, or both even;

$$\therefore p'^2-Nq'^2=A \text{ and } p''^2-Nq''^2=A,$$
$$\text{or } p'^2-Nq'^2=-A, \ ,, \ p''^2-Nq''^2=-A,$$

according as the order of $\frac{p'}{q'}$ is even, or odd.

In the first case, $x=p'$, $y=q'$, $x=p''$, $y=q''$, are both solutions of (1), and (2) has no solution.

In the second case, $x=p'$, $y=q'$, $x=p''$, $y=q''$, are both solutions of (2), and (1) has no solution.

182. When any one solution of $x^2 - Ny^2 = \pm A$ has been found, a series of solutions can be obtained, as follows.

Let h, k any two numbers, such that $h^2 - Nk^2 = 1$, obtained from Art. 178; and let $x = p'$, $y = q'$ be a solution of $x^2 - Ny^2 = \pm A$;

$$\begin{aligned} \therefore\ x^2 - Ny^2 = \pm A &= (h^2 - Nk^2)(p'^2 - Nq'^2), \\ &= (p'h + Nq'k)^2 - N(p'k + q'h)^2, \\ \text{or } &= (p'h - Nq'k)^2 - N(p'k - q'h)^2; \end{aligned}$$

$$\therefore \text{ we can put } \left.\begin{aligned} x &= p'h \pm Nq'k \\ \text{and } y &= p'k \pm q'h \end{aligned}\right\} \quad . \quad . \quad . \quad (1),$$

both upper, or both lower, signs being taken.

Now obtain, from Art. 178, a series of pairs of values for h and k, then for every pair we have from (1) two pairs of values for x and y.

183. *To solve the equation* $\mathrm{x^2 - Ny^2 = \pm B^2A}$; where A is a denominator of a complete quotient occurring in the conversion of $\sqrt{N}$ into a continued fraction.

Put $x = Bx'$, $y = By'$, and we have

$$x'^2 - Ny'^2 = \pm A.$$

These can be solved by Art. 181 or Art. 182.

184. The general equation of the second degree with two symbols, x and y, is

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

The solution of which in positive integers cannot always be effected. We have, however, just been discussing some particular cases of it, and we shall now show how to obtain positive integral solutions when the square of one of the symbols is absent from the equation, or, in other words, when $A = 0$, or $C = 0$.

The method is most simply exhibited by an example.

Ex. Solve in positive integers $3x^2-2xy+x+3y+3=0$.

Here $y=\dfrac{3x^2+x+3}{2x-3}=\dfrac{3}{2}x+\dfrac{11}{4}+\dfrac{45}{4(2x-3)}$.

Clearing of numerical fractions, by multiplying by 4, we have

$$4y=6x+11+\frac{45}{2x-3}.$$

Now x and y being integers, $\dfrac{45}{2x-3}$ must be an integer also;

$\therefore$ $2x-3$ must be a factor of 45;

$$\therefore\ 2x-3=\pm1,\ \pm3,\ \pm5,\ \pm9,\ \pm15,\text{ or }\pm45.$$

Of these only the following give positive integral solutions.

$2x-3=+\ 1$; $\therefore\ x=\ 2$, $4y=12+11+45$; $\therefore\ y=17$.
$2x-3=+\ 3$; $\therefore\ x=\ 3$, $4y=18+11+15$; $\therefore\ y=11$.
$2x-3=+\ 5$; $\therefore\ x=\ 4$, $4y=24+11+\ 9$; $\therefore\ y=11$.
$2x-3=+\ 9$; $\therefore\ x=\ 6$, $4y=36+11+\ 5$; $\therefore\ y=13$.
$2x-3=+15$; $\therefore\ x=\ 9$, $4y=54+11+\ 3$; $\therefore\ y=17$.
$2x-3=+45$; $\therefore\ x=24$, $4y=96+11+\ 1$; $\therefore\ y=27$.

185. For further information on indeterminate equations the reader is referred to Barlow's *Theory of Numbers*, whence the greater portion of this Chapter is derived.

EXAMPLES.—XXXVII.

Obtain one solution in positive integers, and the general form of such solutions, for each of the following equations.

1. $x^2-23y^2=1$. 2. $x^2-17y^2=1$. 3. $x^2-15y^2=1$.
4. $x^2-\ 5y^2=1$. 5. $x^2-14y^2=1$. 6. $x^2-12y^2=1$.
7. $x^2-13y^2=-1$. 8. $x^2-\ 7y^2=2$. 9. $x^2-13y^2=-3$.

Solve in positive integers the following equations.

10. $x^2+2xy-4x-9=0$. 11. $2x^2-xy-3x-y+2=0$.
12. $3xy+2x-7y=3$. 13. $2xy+3y=4x+24$.

Solve when possible the following equations.

14. $x^2-7y^2=-1$. 15. $x^2-17y^2=-1$. 16. $x^2-19y^2=\pm1$.

XVIII

Recurring Series.

186. A *recurring* series is one in which, after some one term, each term is the algebraic sum of the products obtained by taking always the same number of the immediately preceding terms, and multiplying them respectively by certain constants.

Thus $1+3x+2x^2+4x^3-2x^4+16x^5-38x^6+124x^7+\text{etc.}(A)$, is a recurring series, for *after the fourth term* each term is the sum of the products of the preceding term by $-2x$, and of the term before by $3x^2$; so that, if u_n, u_{n-1}, u_{n-2} represent any three consecutive terms, the following *relation* holds between them,

$$u_n = -2xu_{n-1} + 3x^2u_{n-2},$$

$$\text{or} \quad u_n + 2xu_{n-1} - 3x^2u_{n-2} = 0, \qquad . \qquad . \qquad (1).$$

The expression $1+2x-3x^2$ is called the *scale of relation*, being made up of the coefficients of the various terms of the *relation* (1).

Again it will be found that

$$2+x+3x^2-x^3+x^4-10x^5+6x^6-42x^7+\text{etc.}, \qquad (B),$$

is a recurring series, of which the scale of relation is $1+0.x-4x^2+2x^3$, or $1-4x^2+2x^3$.

187. We divide recurring series into different orders according to the number of terms necessary for forming each successive term.

Thus, series (A) is of the second order; series (B) is of the third, for 0 has to be considered as the constant multiplier of the term immediately preceding.

The G.P. $a+ar+ar^2+\text{etc.}$ is a recurring series of the first order, the scale of relation being $1-r$.

Again, in Art. 59, the expansion of $\dfrac{a_0+a_1x}{b_0+b_1x+b_2x^2}$ is a recurring series, of which the scale of relation is $b_0+b_1x+b_2x^2$.

188. Prop. *To find the scale of relation of a recurring series.*

Ex. Let the series be

$$2+5x+2x^2+7x^3+20x^4+61x^5+182x^6+\text{etc.}$$

Assume the scale to be $1-px-qx^2-rx^3$.

Then for determining the constant multipliers p, q, r we have the following relations,

$$\begin{aligned} 5r+\ 2q+\ 7p-\ 20&=0,\\ 2r+\ 7q+20p-\ 61&=0,\\ 7r+20q+61p-182&=0; \end{aligned}$$

which may be obtained by writing down the relations similar to (1) of Art. 186, and dividing by the powers of x which they contain. Solving these, we find $r=0$, $q=3$, $p=2$; $\therefore$ the scale of relation is $1-2x-3x^2$, and the terms begin to recur after the fourth.

Generally, if we know $2m$ or $2m+1$ consecutive terms we can obtain m relations amongst the terms, which will enable us to determine m constant multipliers, and thus we may assume the scale to have $m+1$ terms in all.

189. Prop. *To find the sum of the first* n *terms of a recurring series.*

Denote the series by $u_0+u_1x+u_2x^2+\ \ldots\ +u_mx^m+\text{etc.}$, and the required sum by S.

The following is an *example* of the method to be pursued.

Suppose the scale of relation to be $1-px-qx^2$, and that all terms, after the second, are formed according to this relation, so that, if r is greater than 1, $u_r-pu_{r-1}-qu_{r-2}=0$, . (1).

Now $S = u_0 + u_1x + u_2x^2 + \dots + u_{n-3}x^{n-3} + u_{n-2}x^{n-2} + u_{n-1}x^{n-1}$;

$\therefore\ -pxS = \quad -pu_0x - pu_1x^2 - \dots \quad - \quad -pu_{n-2}x^{n-1} - pu_{n-1}x^n$,

$-qx^2S = \quad -qu_0x^2 - \dots \quad -qu_{n-3}x^{n-1} - qu_{n-2}x^n - qu_{n-1}x^{n+1}$;

$\therefore$, adding,

$$S(1-px-qx^2) = u_0 + (u_1 - pu_0)x - (pu_{n-1} + qu_{n-2})x^n - qu_{n-1}x^{n+1},$$

since the coefficient of x^2, viz., $u_2 - pu_1 - qu_0$ vanishes by (1), as also the coefficient of every power of x up to x^{n-1} inclusive.

But by (1) again $pu_{n-1} + qu_{n-2} = u_n$;

$$\therefore\ S = \frac{u_0 + (u_1 - pu_0)x}{1-px-qx^2} - \frac{u_nx^n + qu_{n-1}x^{n+1}}{1-px-qx^2}.$$

COR. 1. We have

$$\frac{u_0 + (u_1 - pu_0)x}{1-px-qx^2} = u_0 + u_1x + \dots + u_{n-1}x^{n-1} + \frac{u_nx^n + qu_{n-1}x^{n+1}}{1-px-qx^2},$$

or, in other words, if we carry the division of $u_0 + (u_1 - pu_0)x$ by $1-px-qx^2$ to n steps, the quotient is the given series carried on to n terms, whatever number n may be.

Hence $\dfrac{u_0 + (u_1 - pu_0)x}{1-px-qx^2}$ is the *generating function* of the series (Art. 56).

COR. 2. Also the remainder after this division is

$$\frac{u_nx^n + qu_{n-1}x^{n+1}}{1-px-qx^2}.$$

If we give such values to the letters involved that, when n is endlessly increased, this remainder is endlessly decreased, then $\dfrac{u_0 + (u_1 - pu_0)x}{1-px-qx^2}$ is the sum of the series *ad infinitum*. This is one instance in which we can see that the assumption in Art. 64 is true.

COR. 3. We can put the *remainder* into the same form as the generating function, for $qu_{n-1} = u_{n+1} - pu_n$; $\therefore$ the remainder $= \dfrac{u_nx^n + (u_{n+1} - pu_n)x^{n+1}}{1-px-qx^2}$, which becomes $\dfrac{u_0 + (u_1 - pu_0)x}{1-px-qx^2}$ when to n we give the particular value 0.

Similarly, if the scale of relation had been $1-px-qx^2-rx^3$, we should have had

$$S=\frac{u_0+(u_1-pu_0)x+(u_2-pu_1-qu_0)x^2}{1-px-qx^2-rx^3},$$

$$-\frac{u_nx^n+(u_{n+1}-pu_n)x^{n+1}+(u_{n+2}-pu_{n+1}-qu_n)x^{n+2}}{1-px-qx^2-rx^3}.$$

190. Since in order to find the sum of n terms we must know the nth term, we proceed to the

PROP. *To find the general term of a recurring series.*

Consider the same example as in Art. 189.

Let α, β be the roots of the equation $1-px-qx^2=0$;

$$\therefore\ 1-px-qx^2=-q(x-\alpha)(x-\beta);$$

$\therefore$, by partial fractions, the generating function

$$=\frac{A}{\alpha-x}+\frac{B}{\beta-x}=\frac{A}{\alpha}\left(1-\frac{x}{\alpha}\right)^{-1}+\frac{B}{\beta}\left(1-\frac{x}{\beta}\right)^{-1}$$

$$=\frac{A}{\alpha}\left(1+\frac{x}{\alpha}+\ldots+\frac{x^{r-1}}{\alpha^{r-1}}+\text{etc.}\right)+\frac{B}{\beta}\left(1+\frac{x}{\beta}+\ldots+\frac{x^{r-1}}{\beta^{r-1}}+\text{ etc.}\right).$$

But this must be the same as the given series since both are expansions of the same generating function, and it is always possible to give such values to x, including zero, as will make both series convergent (Art. 94);

$\therefore$ the general, or rth, term of the series is $\left(\frac{A}{\alpha^r}+\frac{B}{\beta^r}\right)x^{r-1}$,

where A and B are to be determined by the rules given in the Chapter on Partial Fractions.

Another method of finding the general term is given in Chapter XXIV.

191. Numerical example worked out.

$$2+5x+2x^2+7x^3+20x^4+\text{etc.}$$

(1.) The *scale of relation* has been already found, in Art. 188. It is $1-2x-3x^2$.

(2.) The *generating function.* Let S denote the sum of the first n terms.

$$\begin{array}{lrl} \text{Then} & S= & 2+5x+\ \ 2x^2+\ \ 7x^3+20x^4+\text{etc. to } n \text{ terms,} \\ & -2x\,S= & \quad -4x-10x^2-\ \ 4x^3-14x^4- \quad\ \text{,,} \quad , \\ & -3x^2S= & \qquad\quad -\ \ 6x^2-15x^3-\ \ 6x^4- \quad\ \text{,,} \quad ; \end{array}$$

$$\therefore S(1-2x-3x^2)=2+x-14x^2-12x^3+\text{terms involving } x^n\text{; etc.;}$$

$$\therefore\ S=\frac{2+x-14x^2-12x^3}{1-2x-3x^2}+\text{ remainder.}$$

Hence the generating function is

$$\frac{2+x-14x^2-12x^3}{1-2x-3x^2}=4x+2+\frac{x}{1-2x-3x^2}.$$

(3.) The *general term.* The roots of the equation $1-2x-3x^2=0$ can be found to be $\frac{1}{3}$, and -1, therefore $1-2x-3x^2=(1-3x)(1+x)$;

$$\therefore \text{ we put } \frac{x}{1-2x-3x^2}=\frac{A}{1-3x}+\frac{B}{1+x};$$

$$\therefore\ x=A(1+x)+B(1-3x)$$

$$\therefore \left.\begin{array}{l} A+B=0 \\ A-3B=1 \end{array}\right\}; \ \therefore B=-\tfrac{1}{4},\ A=\tfrac{1}{4};$$

$$\therefore \frac{x}{1-2x-3x^2}=\frac{1}{4}\,\frac{1}{1-3x}-\frac{1}{4}\,\frac{1}{1+x}$$

$$=\tfrac{1}{4}(1+3x+\overline{3x}|^{2}+\ \ldots\ +\overline{3x}|^{r-1}+\text{ etc.})$$

$$-\tfrac{1}{4}(1-x+x^2-\ \ldots\ +\overline{-x}|^{r-1}+\text{ etc.}).$$

Thus the rth term is $\frac{1}{4}\,(3^{r-1}-\overline{-1}|^{r-1})x^{r-1}$.

(4.) The *sum* of n terms, beginning with the first.

$$\begin{array}{rl} S= & 2+5x+\qquad \&\text{c.}+\tfrac{1}{4}\{3^{n-2}-(-1)^{n-2}\}x^{n-2}+\tfrac{1}{4}\{3^{n-1}-(-1)^{n-1}\}x^{n-1}; \\ \therefore -2xS= & \ \ -4x-\qquad \&\text{c.}-\tfrac{2}{4}\{3^{n-1}-(-1)^{n-1}\}x^n, \\ -3x^2S= & \qquad\quad -6x^2-\&\text{c.}-\tfrac{3}{4}\{3^{n-2}-(-1)^{n-2}\}x^n-\tfrac{3}{4}\{3^{n-1}-(-1)^{n-1}\}x^{n+1}; \end{array}$$

$$\therefore S(1-2x-3x^2)=2+x-14x^2-12x^3-\tfrac{1}{4}\{3^n-(-1)^n\}x^n-\tfrac{1}{4}\{3^n+3(-1)^n\}x^{n+1}.$$

We have been at no pains to find the first four terms of this last expression, as we had already obtained them in (2). Also we knew that all the powers of x from x^4 to x^{n-1} inclusive would vanish. We had therefore only to find the co-efficients of x^n and

x^{n+1}, the former of which accidentally came into the unusually neat form of $\frac{3^n-(-1)^n}{4}$, since $3.3^{n-2}+2.3^{n-1}=3^{n-1}+2.3^{n-1}=3^n$, and $-2(-1)^{n-1}-3(-1)^{n-2}=+2(-1)^n-3(-1)^n=-(-1)^n$.

$$\therefore S=\frac{2+x-14x^2-12x^3}{1-2x-3x^2}-\frac{\{3^n-(-1)^n\}x^n+\{3^n+3(-1)^n\}x^{n+1}}{4(1-2x-3x^2)}$$

$$=4x+2+\frac{x}{1-2x-3x^2}-\frac{\{3^n-(-1)^n\}x^n+\{3^n+3(-1)^n\}x^{n+1}}{4(1-2x-3x^2)}$$

It will be observed that the first of these fractions is what the second becomes when in it we put $n=0$.

192. In Art. **190** we showed that the series was equal to the sum of two geometric progressions, of which the first terms were $\frac{A}{\alpha}$, $\frac{B}{\beta}$, and the common factors $\frac{x}{\alpha}$ and $\frac{x}{\beta}$.

It is often asserted that any recurring series can be expressed as the sum of as many geometric progressions as there are simple factors in the scale of relation. This assertion, however, is not true if two or more factors are the same.

For example, in Art. 190, if $\alpha=\beta$, the scale of relation $=-q(x-\alpha)^2$, and we should put, by partial fractions, the generating function $=\frac{A}{\alpha-x}+\frac{B}{(\alpha-x)^2}$, which expands into

$$\frac{A}{\alpha}\left(1+\frac{x}{\alpha}+\frac{x^2}{\alpha^2}+\text{ etc.}\right)+\frac{B}{\alpha^2}\left(1+\frac{2x}{\alpha}+\frac{3x^2}{\alpha^2}+\text{ etc.}\right),$$

of which two series the latter is not a geometric progression.

When we can resolve a series into two, or more, geometric progressions, we can often best find the sum of n terms by obtaining the sum of each progression separately, and then adding together these separate sums. Thus, in Art. 190, the sum of n terms is $\frac{A}{\alpha}\,\frac{\left|\frac{x}{\alpha}\right|^n-1}{\frac{x}{\alpha}-1}+\frac{B}{\beta}\,\frac{\left|\frac{x}{\beta}\right|^n-1}{\frac{x}{\beta}-1}$,

and in Art. 191, from (3),

$$S=\frac{1}{4}\,\frac{\overline{3x}|^n-1}{3x-1}+\frac{1}{4}\,\frac{\overline{-x}|^n-1}{x+1}.$$

193. The scale of relation is sometimes expressed thus, $-p-q$, instead of in the form $1-px-qx^2$.

194. The recurring series $5+9+16+26+39+$etc., can be discussed as follows. Consider the series

$$5+9x+16x^2+26x^3+39x^4+\text{etc.}$$

Find the scale of relation, or whatever may be required, for this series, and then put $x=1$ in the result.

If, however, $x-1$ is a factor of the scale of relation, this method does not give the sum of n terms readily.

EXAMPLES.—XXXVIII.

Find (1) the scale of relation, (2) the generating function, when possible, (3) the general term, (4) the sum of n terms, in each of the following twelve recurring series:—

1. $1+2x+3x^2+4x^3+$etc.
2. $1-3x+5x^2-7x^3+9x^4-$etc.
3. $1+11x+89x^2+659x^3+$etc.
4. $10+14x+10x^2+6x^3+$etc.
5. $1+4+5-2-19-$etc.
6. $1-\frac{5x}{2}+\frac{7}{4}x^2-\frac{17}{8}x^3+$etc.
7. $1-\frac{7}{3}+\frac{17}{9}-\frac{55}{27}+$etc.
8. $1+4x+9x^2+16x^3+25x^4+$etc.
9. $1-3x-9x^4+27x^5+$etc.
10. $3+5x+7x^2+13x^3+23x^4+45x^5+$etc.
11. $2-x+x^3-2x^4+$etc.
12. $1+11+89+659+$etc.

13. Find the nth term of the recurring series,

$$1+3x+7x^2+13x^3+25x^4+51x^5+103x^6+\text{etc.}$$

14. Find the nth terms, and the generating functions, of the recurring series (1.) $2+7x+25x^2+91x^3+\ldots$,

(2.) $2+x+25x^2+37x^3+\ldots$

15. Show that the nth term of the recurring series $2+6+20+72+\ldots$ is $2^{n-1}(1+2^{n-1})$.

16. If a series be formed having for its rth term the sum of r terms of a given recurring series, show that it will also form a recurring series, whose scale of relation will consist of one more term than that of the given series.

Find the scale of relation, the rth term, and the sum of n terms, of the recurring series $1+6+40+288+\text{etc.}$

Show also that the sum of n terms of the series, formed by taking for its rth term the sum of r terms of this series, is

$$\frac{2}{3^2}(2^{2n}-1)+\frac{4}{7^2}(2^{3n}-1)-\frac{5n}{21}.$$

17. Determine the law of the series $2, -3, 4, 4, 24, 56, 152, 360, \ldots$, and find the sum of n terms.

18. The scale of relation of the recurring series

$$1+3x+5x^2+7x^3+\text{etc.}$$

is of the form $\alpha+\beta$; find α and β and the sum of n terms of the series.

19. Find the general term of the series

$$x+x^2+x^4+x^7+x^{11}+\text{etc.}$$

XIX

Summation of Series.

195. DENOTE any series by $u_1+u_2+\ .\ .\ .\ +u_n+\text{etc.}$
Let S_n denote the sum of the first n terms,

$$\text{then} \qquad S_1=u_1, \quad . \quad . \quad . \quad . \quad (1).$$

$$S_2-S_1=u_2, \quad . \quad . \quad . \quad . \quad (2).$$

$$\text{etc.}=\text{etc.}$$

$$S_n-S_{n-1}=u_n, \quad . \quad . \quad . \quad . \quad (n).$$

We have therefore to find the form of S_n, considered as a function of n, such that it can be made to satisfy each of the equations by giving the proper values to n.

We shall show how to do this in several cases.

196. We may remark in passing that, if we can express u_n as the difference of two functions, thus,

$$u_n=v_n-v_{n-1},$$

where v_n and v_{n-1} are the same functions of n and of $n-1$ respectively, we can immediately obtain the form of S_n.

For then we have from equation (n)

$$S_n-S_{n-1}=v_n-v_{n-1};$$

$$\therefore\ S_n-v_n=S_{n-1}-v_{n-1},$$

and from the preceding equation $S_{n-1}-v_{n-1}=S_{n-2}-v_{n-2}$, and so on. Hence S_n-v_n is an expression which retains a constant value, whatever integral value n may have.

Denote it by C; then $S_n-v_n=C$, or $S_n=C+v_n$, and $C=S_1-v_1=u_1-v_1$.

197. Prop. *To find the sum of* n *terms of the series* $u_1+u_2+\ldots+u_n+$ etc., *where* u_n *is a positive integral function of* n.

Ex. $1^3+2^3+3^3+\ldots+n^3+(n+1)^3+$etc.

So that $u_n=n^3$, and is therefore a function of the 3rd degree.

Let S_n denote the sum of n terms.

Put $S_n=A_0+A_1n+A_2n^2+A_3n^3+A_4n^4$, . . . (1).

That is, we assume S_n to be a positive integral function of n of one degree higher than u_n (see *Obs.* below).

Also

$$S_{n-1}=A_0+A_1(n-1)+A_2(n-1)^2+A_3(n-1)^3+A_4(n-1)^4.$$

But $n^3=S_n-S_{n-1}$, hence, subtracting the expression for S_{n-1} from that for S_n, we have

$$\begin{aligned} n^3 &= A_1(n-\overline{n-1})+A_2(n^2-\overline{n-1}|^2)+A_3(n^3-\overline{n-1}|^3) \\ &\qquad +A_4(n^4-\overline{n-1}|^4) \\ &= A_1+A_2(2n-1)+A_3(3n^2-3n+1) \\ &\qquad +A_4(4n^3-6n^2+4n-1), \quad . \quad (2). \end{aligned}$$

Now this is to be true for all positive integral values of n;

$\therefore$, Art. 87, $4A_4=1$, $3A_3-6A_4=0$, $2A_2-3A_3+4A_4=0$,

$$A_1-A_2+A_3-A_4=0;$$

$$\therefore\ A_4=\frac{1}{4},\ A_3=\frac{1}{2},\ A_2=\frac{1}{4},\ A_1=0;$$

$$\therefore\ S_n=\frac{1}{4}n^4+\frac{1}{2}n^3+\frac{1}{4}n^2+A_0, \quad . \quad . \quad (3).$$

To determine A_0 we have $S_1=u_1=1$; $\therefore$ putting $n=1$ in (3),

$$\frac{1}{4}+\frac{1}{2}+\frac{1}{4}+A_0=S_1=1;$$

$$\therefore\ A_0=0;$$

$$\therefore\ S_n=\frac{n^4+2n^3+n^2}{4}=\left(\frac{n(n+1)}{2}\right)^2.$$

Obs. We can see now why we went up to A_4n^4 in S_n, and no further.

If we had stopped at A_3n^3 in (1), we should not have had an

n^3 on the right-hand side of (2), which we must have in order to equate its coefficient with that of n^3 on the left.

If, again, we went as far as A_5n^5 in (1) and stopped, we should have on the right hand of (2) $A_5(5n^4-\text{etc.})$, etc., and when we equated coefficients of n^4 we should have $5A_5=0$, *i.e.* $A_5=0$.

Similarly, if we stopped at A_6n^6, we should have $6A_6=0$; $\therefore A_6=0$, and thence $A_5=0$.

EXAMPLES.—XXXIX.

Sum to n terms the following fifteen series—

1. $1^2+2^2+3^2+4^2+$ etc.
2. $3^2-5^2+7^2-$ etc.
3. $1^3+3^3+5^3+7^3+\ldots+(2n-1)^3+$ etc.
4. $1^3-3^3+5^3-$ etc.
5. $1^4+2^4+3^4+$ etc.
6. $1.3.4+2.4.5+3.5.6+\ldots+n(n+2)(n+3)+$ etc.
7. $1.3-2.4+3.5-$ etc.
8. $1.3.5+2.4.6+3.5.7+$ etc.
9. $a^3+(a+b)^3+\ldots+(a+\overline{n-1}b)^3+$ etc.
10. $1^2+(1^2+2^2)+(1^2+2^2+3^2)+$ etc., each bracket being taken as a term.
11. $a(a+b)(a+2b)+(a+b)(a+2b)(a+3b)+$ etc.
12. $1^2.3+2^2.4+3^2.5+$ etc.
13. $1^3.2+2^3.3+3^3.4+$ etc.
14. $2.4.5+4.6.7+6.8.9+$ etc.
15. $n+2(n-1)+3(n-2)+$ etc.
16. Find the sums of the first n terms of the series of which the nth terms are $(2n+1)(3n-1)$ and $(-1)^{n-1}n^2(2n-1)$.
17. Find the nth term, and the sum of n terms, of the series

$$1.2+2.3+4.5+7.8+11.12+\text{etc.}$$

18. Sum to n terms the series whose nth term is n^2-1.

198. The following is a proof of the general case of Art. **197**.

Let u_n be a positive integral function of n, of the mth degree, say $u_n = A_m n^m + \ . \ . \ . \ + A_1 n + A_0$.

We will show that the equations in Art. **196** will be satisfied by assuming that S_n is a positive integral function of n of the $(m+1)$th degree.

For put $S_n = B_k n^k + \ . \ . \ . \ + B_1 n + B_0$;

then, from equation (n), $A_m n^m + \ldots + A_1 n + A_0 = S_n - S_{n-1}$

$$= B_k\{n^k - \overline{n-1}|^k\} + B_{k-1}\{n^{k-1} - \overline{n-1}|^{k-1}\} + \ldots + B_1,$$

$$= B_k k n^{k-1} + \text{ lower powers of } n, \quad . \quad . \quad . \qquad (1);$$

$\therefore B_k = 0$, if $k-1 > m$ or $k > m+1$.

Hence S_n need not contain a power of n higher than the $(m+1)$th.

From (1), by equating coefficients of like powers of n, we can obtain $m+1$ linear equations for determining $B_{m+1}, B_m, \ . \ . \ . \ B_2, B_1$.

These being found we obtain B_0 from equation (1) of Art. **195**.

199. The particular case, in which each term consists of m *factors* belonging to the *same* A. P., whatever term we consider, can be treated more shortly as follows.

Let $u_n = (b+na)(b+\overline{n+1}a) \ . \ . \ . \ (b+\overline{n+m-1}a)$.

This can be put into the form

$$\frac{(b+na)\ldots(b+\overline{n+m-1}a)(b+\overline{n+m}a)}{a(m+1)}$$
$$-\frac{(b+\overline{n-1}a)(b+n-a)\ldots(b+\overline{n+m-1}a)}{a(m+1)} \qquad (1).$$

Hence we assume at once, Art. **196**,

$$S_n = \frac{(b+na)\ldots(b+\overline{n+m}a)}{a(m+1)} + C,$$

where C does not contain n.

For we see from (1) that this form for S_n satisfies equations (2), . . . (n) of Art. **195**.

By equation (1) of Art. 195 we have

$$(b+a)\ldots(b+ma)=S_1=\frac{(b+a)\ldots(b+ma)(b+\overline{m+1}a)}{a(m+1)}+C$$

$$=(b+a)\ldots(b+ma)\left\{1+\frac{b}{a(m+1)}\right\}+C;$$

$$\therefore C=-\frac{b(b+a)\ldots(b+ma)}{a(m+1)};$$

$$\therefore S_n=\frac{(b+na)\ldots(b+\overline{n+m}a)}{a(m+1)}-\frac{b(b+a)\ldots(b+ma)}{a(m+1)}.$$

Hence the required sum is the difference between two parts, of which *the first* is the nth term of the given series multiplied by a factor next after its greatest, divided by the number of factors so increased and the common difference of the A. P; whilst *the second* part is what the first becomes when in it we put $n=0$.

200. Or we might arrange the proof as follows.

Denote the factors of u_n by $f_n, f_{n+1}, \ldots, f_{n+m-1}$. So that $f_{n+1}-f_n=a$, $f_{n+2}-f_{n+1}=a$, etc., $f_{n+m}-f_{n+m-1}=a$.

Therefore, adding, $f_{n+m}-f_n=m.a$;

$$\therefore f_{n+m}-f_{n-1}=(m+1)a;$$

$$\therefore u_n=f_nf_{n+1}\ldots f_{n+m-1}$$

$$=f_nf_{n+1}\ldots f_{n+m-1}\frac{f_{n+m}-f_{n-1}}{(m+1)a};$$

$$\text{or } u_n=\frac{f_nf_{n+1}\ldots f_{n+m}}{(m+1)a}-\frac{f_{n-1}f_n\ldots f_{n+m-1}}{(m+1)a};$$

$$\therefore u_{n-1}=\frac{f_{n-1}f_n\ldots f_{n+m-1}}{(m+1)a}-\frac{f_{n-2}f_n\ldots f_{n+m-2}}{(m+1)a};$$

etc. $=$ etc.

$$u_1=\frac{f_1f_2\ldots f_{1+m}}{(m+1)a}-\frac{f_0f_1\ldots f_m}{(m+1)a};$$

$\therefore$, adding, and denoting the sum $u_1+u_2+\text{etc.}+u_{n-1}+u_n$ by S_n, we have

$$S_n=\frac{f_nf_{n+1}\ldots f_{n+m}}{(m+1)a}-\frac{f_0f_1\ldots f_m}{(m+1)a}.$$

201. *Ex.* Sum to n terms the series whose nth term is $(4n^2-1)(4n^2-9)$.

Let S_n denote the required sum, then $S_n - S_{n-1} =$ the nth term

$$=(2n-3)(2n-1)(2n+1)(2n+3)$$

$$=\frac{(2n-3)(2n-1)(2n+1)(2n+3)(2n+5)}{10}$$

$$-\frac{(2n-5)(2n-3)(2n-1)(2n+1)(2n+3)}{10}.$$

Assume then

$$S_n=\frac{(2n-3)(2n-1)(2n+1)(2n+3)(2n+5)}{10}+C,$$

where C is the same whatever number of terms we sum.

Put $n=1$, then $\frac{(-1)(1).3.5.7}{10}+C=$1st term$=(-1)(1).3.5$;

$$\therefore\ C=(-3.5)\left(1-\frac{7}{10}\right)=-\frac{9}{2};$$

$$\therefore\ S_n=\frac{(4n^2-1)(4n^2-9)(2n+5)}{10}-\frac{9}{2}.$$

Obs. Although we have verbally expressed the result of Art. 199 in a form easily remembered, we recommend the student not to quote it, but to *work* out each example as above.

202. Many series can be broken up into two or more, each similar to that in Art. 199, and thus their sum can be obtained in the manner *indicated* below.

Ex. Let the nth term be $(n-1)^2(n)(n+1)$.

This $=(n-2+1)(n-1)n(n+1)$

$=(n-2)(n-1)n(n+1)+(n-1)n(n+1)$.

Thus the given series is the sum of two, of which the nth terms are

$$(n-2)(n-1)n(n+1), \text{ and } (n-1)n(n+1).$$

It can be shown as in Art. 201 that the sum

$$\text{of the first} \quad =\frac{(n-2)(n-1)n(n+1)(n+2)}{5},$$

$$\text{and of the second} =\frac{(n-1)n(n+1)(n+2)}{4};$$

$$\therefore \text{ the required sum} =(n-1)n(n+1)(n+2)\left(\frac{n-2}{5}+\frac{1}{4}\right)$$

$$=\frac{1}{20}(n-1)n(n+1)(n+2)(4n-3).$$

EXAMPLES.—XL.

Sum the following six series to n terms.

1. $a(a+b)(a+2b)+(a+b)(a+2b)(a+3b)+$ etc.
2. $2.4+4.6+6.8+$ etc.
3. $1.2.3+2.3.4+3.4.5+$ etc.
4. $1.2.3.4+2.3.4.5+3.4.5.6+$ etc.
5. $1.3.5+3.5.7+5.7.9+$ etc.
6. $2.5.8+5.8.11+8.11.14+$ etc.
7. Prove that $1.2 \ldots p+2.3 \ldots (p+1)+$ etc.

$$+(n-p)(n-p+1) \ldots (n-1)=\frac{\lfloor n}{(p+1)\lfloor n-p-1}.$$

Sum the following six series to n terms.

8. $1.4.3+2.9.4+3.16.5+ \ldots +n(n+1)^2(n+2)$.
9. $1.3+2.4+3.5+ \ldots\ldots +n(n+2)$.
10. $1.3.4+2.4.5+3.5.6+$ etc.
11. $2.5.7+3.6.8+4.7.9+$ etc.
12. $1.4.7+4.7.10+7.10.13+$ etc.
13. $1.2.3.8+2.3.4.9+3.4.5.10+$ etc.

203. The following is another case in which the method of Art. 196 can be applied, and in which the reader will see that, as in Art. 199, the difficulty lies in putting u_n into the form of the difference between two expressions, of which one is the same function of n that the other is of $n-1$.

Let

$$u_n=\frac{1}{(b+na)(b+\overline{n+1}a)\ldots(b+\overline{n+m-2}a)(b+\overline{n+m-1}a)}.$$

Here $u_n=\dfrac{1}{a(m-1)}\left\{-\dfrac{1}{(b+\overline{n+1}a)\ldots(b+\overline{n+m-1}a)}\right.$

$$\left.+\frac{1}{(b+na)(b+\overline{n+1}a)\ldots(b+\overline{n+m-2}a)}\right\}.$$

Hence assume, Art. 196,

$$S_n=-\frac{1}{a(m-1)}\,\frac{1}{(b+\overline{n+1}a)\ldots(b+\overline{n+m-1}a)}+C.$$

Also

$$\frac{1}{(b+a)\ldots(b+ma)}=u_1=S_1=-\frac{1}{a(m-1)}\,\frac{1}{(b+2a)\ldots(b+ma)}+C;$$

$$\therefore\ a(m-1)=-(b+a)+Ca(m-1)(b+a)(b+2a)\ldots(b+ma);$$

$$\therefore\ C=\frac{b+ma}{a(m-1)(b+a)\ldots(b+\overline{m-1}a)(b+ma)}$$

$$=\frac{1}{a(m-1)(b+a)\ldots(b+\overline{m-1}a)}$$

$$S_n=\frac{1}{a(m-1)(b+a)\ldots(b+\overline{m-1}a)}$$

$$-\frac{1}{a(m-1)(b+\overline{n+1}a)\ldots(b+n+\overline{m-1}a)}.$$

Hence the required sum consists of two parts, of which *the second* is the nth term of the given series with the smallest factor of its denominator cut off, divided by the number of factors so decreased and the common difference of the A. P., whilst *the first* part is what the second becomes when in it we put $n=0$.

204. Or we might arrange the proof as follows:

Denote the nth term by $\dfrac{1}{f_n f_{n+1} \cdots f_{n+m-1}}$, where

$$f_{n+1}-f_n=a,\ f_{n+2}-f_{n+1}=a,\ \text{etc.},\ f_{n+m-1}-f_{n+m-2}=a;$$

$$\therefore\ f_{n+m-1}-f_n=(m-1)a;$$

$$\therefore\ u_n=\frac{1}{f_n \cdots f_{n+m-1}}=\frac{1}{f_n f_{n+1} \cdots f_{n+m-1}}\,\frac{f_{n+m-1}-f_n}{(m-1)a}$$

$$=\left(\frac{1}{f_n f_{n+1} \cdots f_{n+m-2}}-\frac{1}{f_{n+1} \cdots f_{n+m-1}}\right)\frac{1}{a(m-1)};$$

$$\therefore\ u_{n-1}=\left\{\frac{1}{f_{n-1} f_n \cdots f_{n+m-3}}-\frac{1}{f_n f_{n+1} \cdots f_{n+m-2}}\right\}\frac{1}{a(m-1)},$$

etc. = etc.

$$u_1=\left\{\frac{1}{f_1 f_2 \cdots f_{m-1}}-\frac{1}{f_2 f_3 \cdots f_m}\right\}\frac{1}{a(m-1)};$$

$\therefore$, adding, and denoting the sum $u_1+u_2+\text{etc.}+u_{n-1}+u_n$ by S_n, we have

$$S_n=\left\{\frac{1}{f_1 f_2 \cdots f_{m-1}}-\frac{1}{f_{n+1} \cdots f_{n+m-1}}\right\}\frac{1}{a(m-1)}.$$

205. *Ex.* Sum to n terms the series whose nth term is $\dfrac{1}{(4n^2-1)(4n^2-9)}$.

Let S_n denote the required sum, then $S_n-S_{n-1}=$ the nth term

$$=\frac{1}{(2n-3)(2n-1)(2n+1)(2n+3)}$$

$$=-\frac{1}{6(2n-1)(2n+1)(2n+3)}+\frac{1}{6(2n-3)(2n-1)(2n+1)}.$$

Assume then $S_n=-\dfrac{1}{6(2n-1)(2n+1)(2n+3)}+C$,

where C is the same whatever number of terms we sum.

Put $n=1$, then $-\frac{1}{6.3.5}+C=1\text{st term}=-\frac{1}{3.5}$;

$$\therefore C=-\frac{1}{3.5}\left(-\frac{1}{6}+1\right)$$

$$=-\frac{1}{18};$$

$$\therefore S_n=-\frac{1}{18}-\frac{1}{6(2n-1)(2n+1)(2n+3)}.$$

The *Obs.* in Art. 201 applies also to Art. 203, 205.

206. The following is an *indication* of the manner in which some series may be solved, by breaking them up into two or more similar to that in Art. 203.

Ex. Sum to n terms the series,

$$\frac{1}{1.5.7}+\frac{1}{3.7.9}+\text{ etc.}$$

The nth term

$$=\frac{1}{(2n-1)(2n+3)(2n+5)},$$

$$=\frac{2n+1}{(2n-1)(2n+1)(2n+3)(2n+5)},$$

$$=\frac{2n-1+2}{(2n-1)(2n+1)(2n+3)(2n+5)},$$

$$=\frac{1}{(2n+1)(2n+3)(2n+5)}+\frac{2}{(2n-1)(2n+1)(2n+3)(2n+5)}$$

Hence the given series is the sum of two, of which the nth terms are

$$\frac{1}{(2n+1)(2n+3)(2n+5)} \text{ and } \frac{2}{(2n-1)(2n+1)(2n+3)(2n+5)}.$$

The sum of the first is

$$-\frac{1}{2.2(2n+3)(2n+5)}+\frac{1}{2.2.3.5},$$

the sum of the second is

$$-\frac{2}{2.3(2n+1)(2n+3)(2n+5)}+\frac{1}{3.1.3.5}.$$

The required sum is the sum of these two

$$= -\frac{3(2n+1)+4}{12(2n+1)(2n+3)(2n+5)} + \frac{3+4}{12.3.5}$$

$$= -\frac{6n+7}{12(2n+1)(2n+3)(2n+5)} + \frac{7}{180}.$$

The sum to infinity is $\frac{7}{180}$.

EXAMPLES.—XLI.

Sum to n terms, and to infinity, the following sixteen series:

1. $\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+$ etc.

2. $\frac{1}{a(a+b)}+\frac{1}{(a+b)(a+2b)}+$ etc.

3. $\frac{1}{2.4.6}+\frac{1}{4.6.8}+\frac{1}{6.8.10}+$ etc.

4. $\frac{1}{6.8}+\frac{1}{8.10}+\frac{1}{10.12}+$ etc.

5. $\frac{1}{1.4.7}+\frac{1}{4.7.10}+$ etc.

6. $\frac{1}{1.5.9}+\frac{1}{5.9.13}+\frac{1}{9.13.17}+$ etc.

7. $\frac{1}{1.2.3.4}+\frac{1}{2.3.4.5}+\frac{1}{3.4.5.6}+$ etc.

8. $\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+$ etc.

9. $\frac{1}{1.3.5.7}+\frac{1}{3.5.7.9}+$ etc.

10. $\frac{1}{1.2}+\frac{1}{2.3}+\ \ldots\ +\frac{1}{n(n+1)}+$ etc.

11. $\frac{1}{2^2-1}+\frac{1}{3^2-1}+\frac{1}{4^2-1}+$ etc.

12. $\frac{1}{1.3.5}+\frac{1}{2.4.6}+$ etc. $+\frac{1}{n(n+2)(n+4)}+$ etc.

13. $\frac{1}{1.3.4}+\frac{1}{2.4.5}+\frac{1}{3.5.6}+$ etc.

14. $\frac{3}{1.2.3}+\frac{5}{2.3.4}+\frac{7}{3.4.5}+$ etc.

15. $\frac{1}{2.3.4}+\frac{2}{3.4.5}+\frac{3}{4.5.6}+$ etc.

16. $\frac{1}{1.3.5}+\frac{2}{3.5.7}+\frac{3}{5.7.9}+$ etc.

17. Sum the series $\frac{1}{2.4.6}+\frac{2}{3.5.7}+\frac{3}{4.6.8}+$ etc., *ad infinitum*.

18. Sum the series $\frac{n}{1.2.3}+\frac{n-1}{2.3.4}+\dots+\frac{1}{n(n+1)(n+2)}$.

19. Sum the series

$$\frac{1(m+2)}{2.3\dots(m+1)}+\frac{2(m+3)}{3.4\dots(m+2)}+\frac{3(m+4)}{4.5\dots(m+3)}+\text{ etc., } \textit{ad inf.}$$

20. Sum to infinity $\frac{1}{\underline{|n}}+\frac{\underline{|2}}{\underline{|n+1}}+\frac{\underline{|3}}{\underline{|n+2}}+$ etc.

21. Sum to n terms the series

$$\frac{3}{1.2.4.5}+\frac{4}{2.3.5.6}+\frac{5}{3.4.6.7}+\dots$$

22. Sum to n terms the series $\frac{1.4}{2.3}+\frac{2.5}{3.4}+\frac{3.6}{4.5}+$ etc.

207. Prop. *To find the sum of the first* n *terms of the series,*

$$u_0+u_1x+u_2x^2+ \textit{etc.} +u_nx^n+ \textit{etc.};$$

where u_n *is a positive integral function of* n, *and the same in form whatever* n *may be.*

We will first illustrate the method by the following example.

Let $S=1+x+3x^2+7x^3+13x^4+\ldots+(\overline{n-1}|^2-\overline{n-1}+1)x^{n-1}$
$+(n^2-n+1)x^n$;

then $Sx=x+x^2+3x^3+7x^4+\ \ldots\ +(\overline{n-2}|^2-\overline{n-2}+1)x^{n-1}$
$+(n^2-3n+3)x^n+(n^2-n+1)x^{n+1}$;

$\therefore\ S(1-x)=1+2x^2+4x^3+6x^4+\ldots+(2n-4)x^{n-1}+(2n-2)x^n$
$-(n^2-n+1)x^{n+1}$;

$\therefore\ S(1-x)x=x+0+2x^3+4x^4+\ldots+(2n-4)x^n+(2n-2)x^{n+1}$
$-(n^2-n+1)x^{n+2}$;

$\therefore\ S(1-x)^2=1-x+2x^2+2x^3+\text{etc.}\ldots+2x^n-(n^2+n-1)x^{n+1}$
$+(n^2-n+1)x^{n+2}$,

$$=1-x+2x^2\frac{1-x^{n-1}}{1-x}-(n^2+n-1)x^{n+1}$$
$$+(n^2-n+1)x^{n+2};$$

$$\therefore S=\frac{1}{1-x}+2x^2\frac{1-x^{n+1}}{(1-x)^3}-\frac{(n^2+n-1)x^{n+1}-(n^2-n+1)x^{n+2}}{(1-x)^2}.$$

Thus by successively multiplying by x and subtracting (or, which is the same thing, by successively multiplying by $1-x$), we have obtained a series forming a G. P., with two terms before and two terms after it.

Obs. 1. It can be seen that the above is a recurring series, of which the scale of relation is $(1-x)^2$.

Obs. 2. If the coefficient of x^n had been a function of n of the 3rd degree, we should have had to multiply three times by $1-x$, and should have had three terms before, and three after, the G. P., and so on generally.

Obs. 3. If x be <1, x^n and n^2x^n are endlessly decreased as n is endlessly increased, and thus the sum to infinity is $\dfrac{1}{1-x}+\dfrac{2x^2}{(1-x)^3}$.

EXAMPLES.—XLII.

Sum to n terms the following nine series, and to infinity those that are convergent :

1. $a+3a^2+7a^3+15a^4+$ etc.

2. $1+3x+7x^2+13x^3+\ .\ .\ .\ +\dfrac{n^3-1}{n-1}x^n+$ etc.

3. $1.2x+2.3x^2+3.4x^3+$ etc.

4. $\dfrac{1}{r}+\dfrac{2}{r^2}+\dfrac{3}{r^3}+$ etc.

5. $\dfrac{1}{e^x}+\dfrac{3}{e^{3x}}+\dfrac{5}{e^{5x}}+$ etc.

6. $\dfrac{1}{3}+\dfrac{2}{3^2}+\dfrac{3}{3^3}+$ etc.

7. $1.64+3.16+5.4+7.1+9.\frac{1}{4}+$etc.

8. $2.8.9x-3.9.10x^2+4.10.11x^3-5.11.12x^4+$ etc.

9. $\dfrac{1.2}{2^2}+\dfrac{2.3}{2^3}+\dfrac{3.4}{2^4}+$ etc.

10. From 2 deduce the sum of the first n terms of the series

$$1-3+7-13+\ .\ .\ .\ +(-1)^n\frac{n^3-1}{n-1}+\text{ etc.}$$

Sum to n terms the following nine series :

11. $2.5.6-4.7.8+6.9.10-$ etc.

12. $1-3.2+5.2^2-7.2^3+$ etc.

13. $3-8.\frac{1}{2}+15.\frac{1}{4}-$ etc. $.\ .\ .\ +(-1)^{n-1}n(n+2)\dfrac{1}{2^{n-1}}$.

14. $1+7x+26x^2+\ .\ .\ .\ +(n^3-1)x^{n-1}+$ etc.

15. $1+3x+5x^2+7x^3+$ etc.

16. $1+\dfrac{2}{2}+\dfrac{3}{2^2}+\dfrac{4}{2^3}+$ etc.

17. $1-\dfrac{3}{2}+\dfrac{5}{2^2}-\dfrac{7}{2^3}+$ etc.

18. $1.2+2.3x+3.4x^2+\ \ldots\ +n(n+1)x^{n-1}$.

19. $1+(1+x)r+(1+x+2x^2)r^2+(1+x+2x^2+3x^3)r^3+\text{etc}\ldots$
$+(1+x+2x^2+\ldots+nx^n)r^n+$ etc.

208. The following is the general case of Art. 207.

Let S denote the sum u_0+u_1x+ etc. $+u_{n-1}x^{n-1}$, . . (1).

Then

$$S(1-x)=u_0+(u_1-u_0)x+\text{etc.}+(u_{n-1}-u_{n-2})x^{n-1}-u_{n-1}x^n \quad (2).$$

Let u_r, the $\overline{r+1}$ th term of (1), be of the mth degree, say

$$u_r=A_mr^m+A_{m-1}r^{m-1}+\ \ldots\ +A_1r+A_0,$$

and $u_{r-1}=A_m\overline{r-1}|^m+A_{m-1}\overline{r-1}|^{m-1}+\ldots+A_1\overline{r-1}+A_0$.

Then u_r-u_{r-1}

$$=A_m(r^m-\overline{r-1}|^m)+A_{m-1}(r^{m-1}-\overline{r-1}|^{m-1})+..+A_1(r-\overline{r-1})$$

$$=mA_mr^{m-1}+\text{ terms containing lower powers of } r+A_1.$$

Hence for any power of x in (2), except x^0 and x^n, the coefficient is a positive integral function of the index, the same in form whatever power we consider, and also its degree is lower by *one* than the degree of the coefficient of any power of x in (1).

If now we multiply again by $1-x$, a similar result will be obtained for all the terms in $S(1-x)^2$ except the first *two* and the last *two*, the first term in the coefficient of x^r being $m(m-1)A_mr^{m-2}$; and so on for each successive multiplication.

Hence after m multiplications the coefficient of every power of x between x^m and x^{n-1} inclusive reduces to $A_m\underline{|m}$, and therefore their terms form a G. P., viz., $A_m\underline{|m}(x^m+x^{m+1}+\ldots+x^{n-1})$, and there are m terms before and m after it.

Cor. Any such series is a recurring series whose scale of relation is $(1-x)^{m+1}$.

For if we multiply by $1-x$ once more, or in other words, multiply the whole original series by $(1-x)^{m+1}$, all the terms

after the first $\overline{m+1}$ disappear. This shows that any term (as $u_r x^r$) of the series, after the first $\overline{m+1}$ terms, is equal to the sum of the products of each of the preceding $\overline{m+1}$ multiplied by the *corresponding term* of the series $\overline{m+1}x-\frac{(m+1)m}{1.2}x^2+$ etc.

By *corresponding term* is here meant that term which will make the product contain x^r exactly.

209. The following are further examples of the method indicated in Art. 196.

Ex. 1. Sum the series $1+\frac{1.2}{n}+\frac{1.2.3}{n(n+1)}+\ldots$ to p terms.

The pth term

$$=\frac{\lfloor p}{n(n+1)\ldots(n+p-2)}=\frac{1}{3-n}\left\{\frac{\lfloor p+1}{n(n+1)\ldots(n+p-2)}-\frac{\lfloor p}{n(n+1)\ldots(n+p-3)}\right\};$$

$$\therefore \text{ we can put } S_p=\frac{1}{3-n}\cdot\frac{\lfloor p+1}{n(n+1)\ldots(n+p-2)}+C.$$

Put $p=2$, we have

$$\frac{1}{3-n}\frac{\lfloor 3}{n}+C=S_2=1+\frac{2}{n};$$

$$\therefore C=1+\frac{2}{n}\left(1-\frac{3}{3-n}\right),$$

$$=1-\frac{2}{3-n};$$

$$\therefore S_p=\frac{1}{3-n}\frac{\lfloor p+1}{n(n+1)\ldots(n+p-2)}+1-\frac{2}{3-n}.$$

Ex. 2. Show that

$$\frac{1}{p+1}-\frac{r}{(p+1)(p+2)}+\frac{r(r-1)}{(p+1)(p+2)(p+3)}-\text{etc.}=\frac{1}{p+r+1},$$

when r is a positive integer.

Let S_n denote the sum of n terms.

Then $S_n-S_{n-1}=n$th term

$$=(-1)^{n+1}\frac{r(r-1)\ .\ .\ .\ (r-n+2)}{(p+1)(p+2)\ .\ .\ .\ (p+n)}.$$

Assume $S_n=(-1)^{n+1}A\dfrac{r(r-1)\ .\ .\ .\ (r-n+1)}{(p+1)\ .\ .\ .\ (p+n)}+C;$

$$\therefore\ (-1)^{n+1}A\frac{r(r-1)...(r-n+1)}{(p+1)...(p+n)}-(-1)^n\frac{Ar...(r-n+2)}{(p+1)...(p+n-1)},$$

$$=(-1)^{n+1}\frac{r(r-1)\ .\ .\ .\ (r-n+2)}{(p+1)\ .\ .\ .\ (p+n)};$$

$$\therefore A\left\{\frac{(r-n+1)}{p+n}+1\right\}=\frac{1}{p+n};$$

$$\therefore A=\frac{1}{r+p+1},$$

and $S_n=\dfrac{(-1)^{n+1}}{r+p+1}\ \dfrac{r(r-1)\ .\ .\ .\ (r-n+1)}{(p+1)\ .\ .\ .\ (p+n)}+C.$

Put $n=1$ and we have

$$\frac{1}{r+p+1}\cdot\frac{r}{p+1}+C=\frac{1}{p+1};$$

$$\therefore\ C=\frac{1}{p+1}\left(-\frac{r}{r+p+1}+1\right)=\frac{1}{r+p+1};$$

$$\therefore\ S_n=\frac{(-1)^{n+1}}{r+p+1}\cdot\frac{r(r-1)\ .\ .\ .\ (r-n+1)}{(p+1)\ .\ .\ .\ (p+n)}+\frac{1}{r+p+1}.$$

If r be a positive integer, let the nth be the last term;

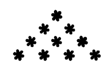
$$\therefore\ r-n+2=1,\ \text{and}\ r-n+1=0;$$

$$\therefore\ S_n=\frac{1}{r+p+1}.$$

210. *Polygonal Numbers.*

The following are specimens of a class of problems with which the student will sometimes meet.

(1.) To find the number of cannon balls, which can be laid out, so as to form an equilateral triangle having n balls in an outside row.



It will be seen from the diagram that each ball of any one row is to fit in between two balls of the next row; thus, beginning from one outside row, each row will contain one ball less than the preceding. Hence the total number of balls is

$$n+\overline{n-1}+\overline{n-2}+\ .\ .\ .\ +3+2+1=\frac{n(n+1)}{2}.$$

(2.) To find the number of balls which can be placed in pyramid, in which the base is an equilateral triangle having n balls in a side. It is evident that the outside row of any layer contains one ball less than the outside row of the layer immediately below.

Hence in the rth layer, counting from the ground, the outside row contains $n-r+1$ balls, and the whole layer, by (1.), $\frac{(n-r+1)(n-r+2)}{2}$. Hence the total number of balls is

$$\frac{(n+1)n}{2}+\frac{n(n-1)}{2}+\ .\ .\ .\ +\frac{3.2}{2}+\frac{2.1}{2}$$

$$=\tfrac{1}{6}(n+2)(n+1)n, \text{ by Art. 199.}$$

(3.) To find the number of balls in a complete pile, of which the base is a rectangle containing m balls in one side and n in another. Suppose m not greater than n.

The rth layer, counting from the ground, will have $m-r+1$ and $n-r+1$ balls in its sides, and therefore will contain $(m-r+1)(n-r+1)$ balls.

The mth layer will consist of a single row containing $n-m+1$ balls. Hence the total number of balls is

$$(n-m+1)+2(n-m+2)+3(n-m+3)+\ .\ .\ .\ +mn$$
$$=(n-m)(1+2+3+\ .\ .\ .\ +m)+1+2^2+3^2+\ .\ .\ .\ +m^2$$
$$=\frac{1}{2}(n-m)m(m+1)+\frac{m(m+1)(2m+1)}{6}$$
$$=\frac{m(m+1)}{6}(3n-3m+2m+1)$$
$$=\frac{m(m+1)}{6}(3n-m+1).$$

EXAMPLES.—XLIII.

1. The sum of the series

$$\frac{4}{1.5}+\frac{9}{5.14}+\frac{16}{14.30}+\frac{25}{30.55}+\ .\ .\ .\ \text{to } n \text{ terms is}$$
$$1-\frac{6}{(n+1)(n+2)(2n+3)},$$

the last factor in the denominator of each term being the sum of the first factor and the numerator.

2. Find the sum of n terms of the series

$$\frac{1}{(a+2b+3c)(2a+3b+4c)}+\frac{1}{(2a+3b+4c)(3a+4b+5c)}+\text{ etc.}$$

3. Sum to n terms the series

$$1.1^2+2(2^2+1^2)+3(3^2+2^2+1^2)+\text{ etc.}$$

4. Find the number of balls in an incomplete pyramid, containing r layers, and having n balls in the side of the lowest layer, which forms an equilateral triangle.

5. Find the number of balls in an incomplete pyramid of 5 layers, the bottom layer being a rectangle having 7 and 9 balls in its sides.

6. Find the number of balls in an incomplete rectangular pile of 18 courses, having 56 balls in the length, and 38 in the breadth, of the base.

7. Find the number of balls in a square pile having 20 in a side of its base.

8. Sum the series $\frac{1}{1.2}+\frac{1.3}{1.2.3.4}+\frac{1.3.5}{1.2.3.4.5.6}+$ etc.

9. Show that the sum of the series

$1+2^2+3+4^2+5+6^2+\ \ldots$ to x terms is $\frac{x+1}{12}(2x^2+x+3)$,

when x is odd.

10. Sum to n terms the series

$$\frac{4}{2.3.4}+\frac{7}{3.4.5}+\frac{10}{4.5.6}+\text{ etc.}$$

11. Sum $1-\frac{3}{4}+\frac{3.5}{4.8}-\frac{3.5.7}{4.8.12}+$ etc., to infinity.

12. Find the sum of the series

$$\frac{1}{1.2}-\frac{x}{2.3}+\frac{x^2}{3.4}-\ \ldots, \text{ to infinity, } x \text{ being } <1.$$

13. Show that, if $S_m=1+\frac{1}{2}+\frac{1}{3}+$ etc. $+\frac{1}{m}$, the sum of n terms of the series

$$\frac{1}{2}\frac{1}{S_1S_2}+\frac{1}{3}\frac{1}{S_2S_3}+\text{ etc. } \ldots +\frac{1}{n+1}\frac{1}{S_nS_{n+1}}$$

is $1-\frac{1}{S_{n+1}}$.

14. Sum to infinity the series $\frac{1}{\lfloor n}+\frac{\lfloor 2}{\lfloor n+1}+\frac{\lfloor 3}{\lfloor n+2}+$ etc.; and prove that $\frac{1}{\lfloor n}=\frac{1}{\lfloor n-1}-\frac{1}{\lfloor 2\,\lfloor n-2}+\frac{1}{\lfloor 3\,\lfloor n-3}-$ etc. to n terms.

15. Sum the series

$n(n+3)+(n-1)(n+4)x+(n-2)(n+5)x^2+$ etc. to n terms.

XX

Theory of Numbers.

211. In this chapter we shall make the following limitations as to the terms employed.

(1.)	The word	*number*	will mean	a *positive integral* number only.
(2.)	„	*divisor*	„	an *exact* divisor only, *i.e.*, a *factor*, see [Art. 134, Note].
(3.)	„	*divisible*	„	divisible *without a remainder*.
(4.)	„	*divides*	„	divides „ „

212. *Def.* A number, which is divisible by no other except itself and unity; is called *a prime number*, or simply, *a prime.*

Def. A number, which is divisible by some other besides itself and unity, is called a composite number.

Thus 6 is a composite number, being divisible by 2 and 3, and can be expressed by the product 2×3; but 5 is a prime, and can only be expressed by a product thus, 1×5.

And, generally, a *composite* number can be expressed by the product of two, or more, factors, neither of which is the number itself or unity; whilst a *prime* cannot be expressed as the product of any two factors besides itself and unity.

It will be observed that 1 is a prime and divides every number.

213. *Def.* Two numbers, which have no common factor except unity, are said *to be prime to one another*, and each is said to be *prime to the other.*

Thus 6 is prime to 25; but 6 and 10 are not prime to one another, having the common factor 2.

The only numbers, to which a prime number is not prime, are its multiples and unity. Thus 7 is prime to all numbers except 1, 7, 14, 21, 28, 35, etc. In other words, a prime number is prime to all except those, of which it is a divisor, and unity.

214. *Def.* When any number is expressed by the continued product of the highest powers of the various primes which will divide it, we say that it is *decomposed*, or *resolved, into its simple factors.*

Thus 15435, when expressed by $3^2.5.7^3$, is decomposed into its simple factors. For $3^2.5.7^3=15435$, and the only primes which will divide 15435 are 3, 5, 7, and 3^2, 5, 7^3 are the highest powers of these primes which will do so.

Ex. Resolve 22869 into its simple factors. On trial we find that

22869 is divisible by 3,
7623 „ 3,
2541 „ 3,
847 „ 7,
121 „ 11;

3	22869
3	7623
3	2541
7	847
11	121
	11

$$\therefore\ 22869=3^3.7.11^2.$$

EXAMPLES.—XLIV.

Resolve the following into their simple factors:—

1. 225.	2. 1023.	3. 289.	4. 4095.	5. 504.
6. 42237.	7. 2628.	8. 271469.	9. 5880.	
10. 1764.	11. 1665.	12. 5670.	13. 30527.	
14. 484000.	15. 2880.	16. 16200.	17. 99225.	

215. Hence any number N can be expressed thus, $N=a^p.b^q.c^r$ . . ., where a, b, c, etc. are all primes and all different, and a^p, b^q, etc. are their severally highest powers which occur in N. This is called the *composition of N.*

We now proceed to prove that a number can be decomposed in only one way, or, as it is sometimes expressed, a number has only one composition.

Thus 22869, being equal to $3^3.7.11^2$, cannot be equal to the product of any other powers of the primes 3, 7, 11, nor to a product containing any other primes but 3, 7, 11, *i.e.*, 22869 cannot be equal to $3^2.7^2.11^3$, or $3^3.5^2.11$, etc.

This is evident in each individual case; but in order to prove it once for all, for all numbers, we must first establish two propositions, Art. 216, 217.

216. In the following Article we shall require the process of finding the G. C. F. of two numbers.

The process was exhibited in [Art. 128] as far as 3 steps. We shall now draw the student's attention to such cases as may not terminate so soon.

Let a and c be the two numbers, and suppose $a>c$.

Divide a by c, let q_1 be quotient and r_1 remainder, $\therefore a=cq_1+r_1$,

" c " r_1 " q_2 " r_2 " $c=r_1q_2+r_2$.

Now at the next step, in [Art. 128], the division was supposed to be exact, but if it is not so, we continue as follows :—

Divide r_1 by r_2, let q_3 be quotient and r_3 remainder, $\therefore r_1=r_2q_3+r_3$,

" r_2 " r_3 " q_4 " r_4 " $r_2=r_3q_4+r_4$;

and we should carry on the process in the same way, till we obtained a remainder (r) which would exactly divide the preceding remainder, and then r can be shown, as d was in [Art. 128], to be the G. C. F. of a and c.

Now if a is prime to c, they can have no C. F. but 1, *i.e.*, their G. C. F. is 1, and $\therefore$ the last remainder (r) must be 1.

217. PROP. *If a number* c *divides* ab, *the product of two numbers* a *and* b, *and is prime to* a, *it must divide* b.

Perform the operation of finding the G. C. F. of a and c, then since a is prime to c, we must at some one step have unity for a remainder.

Let a be $>c$, let the quotients be $q_1, q_2, \ldots q_{n-1}, q_n$,
and the remainders $r_1, r_2, \ldots r_{n-1}, 1$,

then $a=cq_1+r_1$; $\therefore\ ba=bcq_1+br_1$, . . . (1),

$c=r_1q_2+r_2$; $bc=br_1q_2+br_2$, . . . (2),

$r_1=r_2q_3+r_3$; $br_1=br_2q_3+br_3$, . . . (3),

etc. = etc. etc. = etc.

$r_{n-2}=r_{n-1}q_n+1$; $br_{n-2}=br_{n-1}q_n+b$, . . . (n).

From (1), since c divides ba and bc, it divides br_1.

" (2) " bc " br_1, " br_2.

" (3) " br_1 " br_2, " br_3.

Proceeding in this way we can show that c divides the product of b into each remainder; but the last remainder is 1; $\therefore$ c divides $b\times 1$, or b.

The same result would follow if a were $<c$.

COR. 1. *If* a *be a prime*, a^n *is divisible by no prime except* a. For let c be any other prime, and *suppose* it will divide a^n, *i.e.*, it divides $a.a^{n-1}$; $\therefore$, being prime to a, by the Prop. it divides a^{n-1}; in the same way we can show that c must divide each of the powers of a down to a itself inclusive; but, being prime to a, it cannot divide a. Hence the supposition that it could divide a^n was absurd.

COR. 2. *If* c *is prime to* a *and to* b, *it is prime to* a.b. For suppose c has a factor, which will divide ab, this factor cannot divide a, for c, being prime to a, has no factor which will divide a; $\therefore$ by the Prop. this factor must divide b; but this it cannot do, since c is prime to b. Therefore c and ab have no common factor; $\therefore$ c is prime to $a.b$.

Obs. This of course includes the case of c being a prime, and dividing neither a nor b, and $\therefore$ not dividing $a.b$.

218. Prop. *A number can be decomposed into simple factors in only one way.*

Thus, if $N=a^p.b^q.c^r \ldots$, (1), and also $N=\alpha^x.\beta^y.\gamma^z \ldots$, (2),
where $a, b, c, \ldots$ are all primes, and all different, and
$\alpha, \beta, \gamma, \ldots$ " " ;
then α must be equal to one of the primes $a, b, c, \ldots$, say a, and then x must be equal to p.

For if α is not equal to one of the primes $a, b, c, \ldots$, it cannot divide any of their powers $a^p, b^q, \ldots$, (Art. 217, Cor. 1); $\therefore$ it cannot divide the product $a^p.b^q. \ldots$, (Art. 217, *Obs.*).

But from (2) α divides N, and $\therefore$ from (1) it divides the product $a^p.b^q \ldots$

These results contradict one another; $\therefore$ α is equal to one of the primes $a, b, c, \ldots$, let $\alpha=a$.

Similarly each of the primes β, γ, etc. is equal to one of the primes b, c, etc., say $\beta=b$, $\gamma=c$, etc.

Further, if $x>p$, we have

$$b^q.c^r. \ldots = a^{x-p}.b^y.c^z \ldots$$

Here a divides the right-hand side; but, since it is prime to b, c, etc., it cannot divide the left, and this is absurd; $\therefore$ x is not greater than p.

Similarly it can be shown that p is not greater than x; $\therefore$ $x=p$. Similarly $y=q$, etc.

219. If a number is a perfect square, it is the product of two numbers exactly alike, and therefore p, q, r, etc. are all even, and $a^pb^qc^r \ldots$ is the product of $a^{\frac{p}{2}}b^{\frac{q}{2}}c^{\frac{r}{2}} \ldots \times a^{\frac{p}{2}}b^{\frac{q}{2}}c^{\frac{r}{2}} \ldots$ Also N is not a square unless p, q, r, etc. are all even.

220. *Ex.* Find the least number the product of which by 2250 will be a square number.

We have $2250=5^3.3^2.2$, and $5^3.3^2.2\times5.2=5^4.3^2.2^2$.

Now this last is a square number, since all its indices are even; $\therefore$ the required number is $5.2=10$.

EXAMPLES.—XLV.

1. Find the least number the product of which with 1500 will be a perfect square.

2. Find the least number the product of which with 1500 will be a perfect cube.

3. Find the least number the product of which with 14175 will be a perfect cube.

4. Find the least number the product of which with 1323 will be a perfect fourth power.

221. Prop. *To find the various divisors*, or factors, *of a number.*

Denote the number by N, and its composition by $a^p.b^q.c^r \ldots$

As in Art. 218 it is easily seen that no number can divide N unless it be wholly *composed* of two or more of the following numbers,

$$\begin{array}{l} 1,\ a,\ a^2,\ a^3,\ \ldots\ ,\ a^p, \\ 1,\ b,\ b^2,\ \ldots\ldots\ ,\ b^q, \\ 1,\ c,\ \ldots\ldots\ldots\ ,\ c^r, \\ \qquad\qquad \text{etc.,} \end{array}$$

multiplied together, *i.e.*, that it must be a term of the product,

$$(1+a+a^2+\ \ldots\ +a^p)(1+b+\ \ldots\ +b^q)\ \ldots\ ;$$

and that every term of this product is a factor of N.

Cor. 1. The *sum* of the factors of N is this product, and

$$\therefore = \frac{a^{p+1}-1}{a-1} \cdot \frac{b^{q+1}-1}{b-1} \ldots$$

Cor. 2. The *number* of the factors of N is the number of terms in this product, and $\therefore = (p+1)(q+1) \ldots$

If N be not a square number, one at least of p, q, r, etc. is odd, and one of the factors $p+1$, etc. is even, and therefore the number of factors of N is even.

Cor. 3. *To find the number of ways in which* N *can be resolved into two factors.*

Each of the factors of N must be paired with another factor, such that the product of the two may be N; thus we have one way of resolution for each pair of factors. Hence, if N is not a square number, the number of ways required is half the number of factors of N, and $\therefore =\frac{1}{2}(p+1)(q+1)\ldots$

If, however, N is a square number, one way is formed by *repeating* one of the factors, viz., $\sqrt{N}$, and multiplying it into itself, hence the number of ways required is half the number of factors of N *increased by unity*, and

$$\therefore =\frac{(p+1)(q+1)\ldots+1}{2}.$$

222. *Ex.* $144=3^2.2^4$.

Its factors are
$$\left.\begin{matrix} 1, & 2, & 2^2, & 2^3, & 2^4, \\ 3, & 3.2, & 3.2^2, & 3.2^3, & 3.2^4, \\ 3^2, & 3^2.2, & 3^2.2^2, & 3^2.2^3, & 3^2.2^4, \end{matrix}\right\} \quad . \quad . \quad (A),$$
and their number is $(2+1)(4+1)=15$.

The ways in which it can be resolved into two factors are
$1\times 3^2.2^4$, $2\times 3^2.2^3$, $2^2\times 3^2.2^2$, $2^3\times 3^2.2$, $2^4\times 3^2$,
3×3.2^4, 3.2×3.2^3, and $3.2^2\times 3.2^2$.

Thus 3.2^2 has to be multiplied into itself to make up 144; hence in pairing the factors together we bring in another, 3.2^2, to pair with the one we have already, and then half the number of factors so increased is the number required.

223. *Required the number of ways in which a number* (N) *can be resolved into two factors, so that one factor of any pair is prime to the other.*

Denote the composition of N by $a^p.b^q.c^r\ldots$

Now in each pair one factor must contain a^p, and the other factor cannot contain a at all, for otherwise the factors would not be prime to one another; similarly for b^q, etc. Thus, we shall have just the same number of ways, to whatever powers a, b, etc. are raised. Hence the number of ways required is the

same as the number of ways in which the number $a.b.c \ldots$ can be resolved into two factors, and $\therefore = \frac{(1+1)(1+1) \ldots}{2} = 2^{n-1}$, n being the number of the primes a, b, c, etc.

EXAMPLES.—XLVI.

Find the various factors of each of the numbers,

1. 225. 2. 42237. 3. 1764. 4. 48400.

See *Examples* XLIV. 1, 6, 10, 14.

Find also their number and sum; and the number of ways in which each of the above integers can be resolved into two factors, and into two prime to one another.

5. Find the sum of all numbers less than a number and prime to it.

224. Let a be a given number, b any number.

Divide b by a, let q be the quotient, and r the remainder;

$$\therefore b = qa + r, \text{ or } b - r = qa, \quad . \quad . \quad (1).$$

If $b < a$, $q = 0$; if b be a multiple of a, $r = 0$.

Hence any number can be expressed in the form $qa+r$, where q is zero or some positive integer, and r is zero or one of the integers $1, 2, \ldots q-2, q-1$; this limitation will always be supposed to be placed on the symbols q and r, or whatever stand in their places, whenever a number is expressed in this form.

225. Foreign writers have a different phraseology at this point. Instead of saying

b on division by a gives a remainder r, they say

b to modulus a is congruent to r,

and instead of expressing this fact algebraically, as in (1), they write

$$b \equiv r \text{ (mod. } a\text{), or } b - r \equiv 0 \text{ (mod. } a\text{)}, \quad . \quad (2),$$

it being very seldom important to mention the quotient q.

Each of the modes of expression in (2) is called a *congruence;* and all numbers, which on division by a give a remainder r, are said to be congruent to each other to modulus a.

226. *Note.* $4m+3=4(m+1)-1$, which is of the form $4m-1$.

Similarly, if r exceeds $\frac{a}{2}$, $qa+r=a(q+1)-\overline{a-r}$.

Any even number is of the form $2n$.
Any odd „ $2n+1$.

227. *Ex.* 1. Every square number is of one of the forms $5m$, $5m\pm1$.

Every number is of one of the forms $5n$, $5n\pm1$, $5n\pm2$.

Now $(5n)^2=5.(5n^2)$ which is of the form $5.m$,

$(5n\pm1)^2=5.(5n^2\pm2n)+1$ „ $5.m+1$,

$(5n\pm2)^2=5.(5n^2\pm4n+1)-1$ „ $5.m-1$.

Ex. 2. If n is an odd number, $(n^2+3)(n^2+7)$ is divisible by 32.

Since n is odd it is of the form $2m+1$;

$$\therefore (n^2+3)(n^2+7)=(4m^2+4m+4)(4m^2+4m+8)$$
$$=16(m^2+m+1)(m^2+m+2).$$

Now m^2+m+1, m^2+m+2 are two successive numbers, and $\therefore$ one or other must be even, *i.e.*, divisible by 2;

$$\therefore (n^2+3)(n^2+7) \text{ is divisible by } 32.$$

It is easily seen that it is m^2+m+2 which is even, for it $=m(m+1)+2$. Now $m(m+1)$ must be even, since m and $m+1$ are successive integers.

Ex. 3. Every prime greater than 3 is of the form $6m\pm1$.

For every number is of the form $6m$, $6m\pm1$, $6m\pm2$, and $6m+3$.

Now $6m$, $6m\pm2$, are divisible by 2, and $6m+3$ is divisible by 3; $\therefore$ these cannot represent primes; $\therefore$ primes >3 can only be represented by $6m\pm1$.

EXAMPLES.—XLVII.

1. If n be odd, $n(n^2-1)$ is divisible by 24.

2. The difference between any number and its square is even.

3. Prove that the sum and difference of any two odd, or of any two even, numbers are even.

4. The sum of any three odd numbers is odd.

5. Any power of an even number is even, and of any odd number, odd.

6. Every prime number greater than 2 is of the form $4n\pm1$.

7. Every odd square number is of the form $4n+1$, and also of the form $8n+1$.

8. Every even square number is of the form $4n$.

9. The sum of two odd squares cannot be a square.

10. The difference between any two odd squares is divisible by 8.

11. If the sum of two squares is another square, one of the three is divisible by 5.

12. Show that every cube number is of one of the forms $7n$, $7n\pm1$.

13. Show that $n^4-4n^3+5n^2-2n$ is divisible by 12 for all integral values of n above 2.

14. If n be a prime greater than 3, then either n^2+n-2 or n^2-n-2 is divisible by 18.

15. The difference of the squares of any two prime numbers greater than 3 is divisible by 24.

16. Show that to modulus 6 every number is congruent to its cube.

17. The sum of the cubes of two whole numbers, one next greater and one next less than a multiple of 3, is divisible by 36.

18. Find the form of r in order $r(r-1)$ may, when divided by 7, give an odd quotient.

19. Prove that n^5-n is always divisible by 30, and if n is odd by 120.

20. If $n-2$, $n+2$ be both prime numbers greater than 5, prove that n, when divided by 30, will leave a remainder 9, 15, or 21.

21. Find a series of square numbers which, when divided by 7, leave a remainder 4.

22. If n be an even number, n^3+20n is divisible by 48.

23. Show that $(2n+1)^{2m}-1$ is divisible by 8.

24. If a cube be divided by 9, the remainder is 0, 1, or 8.

25. If $a^2+b^2=c^2$, then abc is divisible by 60.

26. The twelfth power of a number is of the form $13n$, or $13n+1$.

228. Prop. *If* a *is prime to* b, *and if the same remainder is left after dividing, by* a, *two numbers of the form* mb+k, m'b+k, *then* m *and* m' *differ by a multiple of* a.

Let r be the remainder, q, q' the quotients; then

$$mb+k=qa+r, \quad m'b+k=q'a+r;$$
$$\therefore (m-m')b=(q-q')a.$$

Hence a divides $(m-m')b$; but it is prime to b; $\therefore$ it divides $m-m'$; in other words, $m-m'$ is a multiple of a. Q.E.D.

Cor. 1. No two of the numbers $0, 1, 2 \ldots a-1$ differ by a multiple of a, each being less than a; $\therefore$ if the numbers

$$k,\ b+k,\ 2b+k,\ \ldots\ \overline{a-1}b+k, \qquad (1),$$

be divided by a, the remainders are all different, and their number being a, and each less than a, they must be the numbers

$$0,\ 1,\ 2,\ 3,\ \ldots\ \overline{a-2},\ \overline{a-1}, \qquad (2),$$

though, of course, not necessarily occurring in this natural order.

Cor. 2. Let $nb+k$ be any one of the numbers in (1), divide it by a, let r be the remainder, and q the quotient; so that $nb+k=qa+r$.

Now if r and a have a common factor it must divide $nb+k$;

therefore amongst the numbers in (1) there are as many as amongst those in (2), which have a factor in common with a.

Hence the number of those which are prime to a in (1) and in (2) is the same, or, in other words, the number of integers in (1) which are prime to a, is the same as the number of integers less than a and prime to it.

229. We use the symbol $P(a)$ to denote the number of integers (including 1) less than a and prime to it.

230. Prop. *If* a *and* b *are two numbers, prime to one another, required* P(a.b), i.e., *required the number of integers less than the product* a.b *and prime to it.*

The first ab numbers can be arranged thus:—

$$\begin{array}{ccccc}
1, & 2, & 3,\ldots & k,\ldots & b, \\
b+1, & b+2, & b+3,\ldots & b+k,\ldots & b+b, \\
2b+1, & 2b+2, & 2b+3,\ldots & 2b+k,\ldots & 2b+b, \\
.\ . & .\ . & .\ .\ ,\ldots & .\ .\ ,\ldots & .\ . \\
.\ . & .\ . & .\ .\ ,\ldots & .\ .\ ,\ldots & .\ . \\
.\ . & .\ . & .\ .\ ,\ldots & .\ .\ ,\ldots & .\ . \\
(a-1)b+1, & (a-1)b+2, & (a-1)b+3.. & (a-1)b+k.. & (a-1)b+b.
\end{array}$$

We will now examine these numbers, and reject those which have a factor in common with a or b.

Consider any one column, *e.g.*, the kth. If k be prime to b, each number in the column is prime to b; but if k and b have a common factor other than 1, it must divide each number in the column, and, therefore, the whole column must be rejected.

Now in the first line there are $P(b)$ numbers prime to b.

Hence there are $P(b)$ columns of numbers prime to b.

Let the kth be one of these. Then (Art. 228, Cor. 2) the number of integers in it prime to a is $P(a)$.

Hence each of the $P(b)$ columns of integers prime to b contain $P(a)$ which are prime to a.

Hence amongst the first ab natural numbers there are $P(a).P(.b)$ which are prime to $a.b$;

$$\therefore P(a.b) = P(a).P(b).$$

COR. 1. Evidently $P(2)=1$.

Let M be any *odd* number, then 2 is prime to M;

$$\therefore P(2M)=P(2).P(M)$$
$$=P(M).$$

COR. 2. Let $a, b, c, \ldots h, k$ be a set of numbers in which each is prime to each, and therefore to the product of any, of the others. Then

$$P(a.b.c \ldots h.k)=P(a)P(b.c \ldots h.k)$$
$$P(b.c \ldots h.k)=P(b)P(c \ldots h.k)$$
$$\text{etc.}=\text{etc.}$$
$$P(h.k)=P(h).P(k)$$

$\therefore$, multiplying, $P(a.b.c \ldots h.k)=P(a).P(b).P(c) \ldots P(h).P(k)$.

231. We will apply this formula to prove the well-known PROP. *To find the number of integers less than a given number and prime to it.*

Let N denote the given number, $a^pb^qc^r \ldots$ its composition.

Amongst the natural numbers $1, 2, 3, \ldots a^p-1, a^p$, the only ones not prime to a are the various multiples of a up to a^p, or $a, 2a, 3a, \ldots a^{p-1}, a^p$, *i.e.*, the numbers which are the products of a into each of the numbers $1, 2, 3 \ldots a^{p-1}$, and $\therefore$ their number is a^{p-1};

$$\therefore P(a^p)=a^p-a^{p-1}=a^p\left(1-\frac{1}{a}\right).$$

Each of the factors $a^p, b^q, c^r, \ldots$ being prime to each of the others, we have, Art. 230, COR. 2,

$$P(N)=P(a^p).P(b^q).P(c^r) \ldots$$
$$=a^p\left(1-\frac{1}{a}\right).b^q\left(1-\frac{1}{b}\right).c^r\left(1-\frac{1}{c}\right) \ldots$$
$$=a^p.b^q.c^r \ldots \left(1-\frac{1}{a}\right)\left(1-\frac{1}{b}\right)\left(1-\frac{1}{c}\right) \ldots$$
$$=N\left(1-\frac{1}{a}\right)\left(1-\frac{1}{b}\right)\left(1-\frac{1}{c}\right) \ldots$$

Note.—Here 1 has been considered as a number prime to ab. If not, we should have as our result

$$N\left(1-\frac{1}{a}\right)\left(1-\frac{1}{b}\right)\left(1-\frac{1}{c}\right)\ldots-1.$$

EXAMPLES.—XLVIII.

1. Find the number of integers less than 360 and prime to it.
2. " " 1023 "
3. " " 1764 "
4. " " 1000 "
5. Show that if any number of square numbers be divided by a given number M, the greatest possible number of different remainders is $\frac{M}{2}+1$.
6. The sum of the numbers less than a given number N, and prime to it $=\frac{N}{2}\times$ the number of numbers less than N and prime to it.

232. Prop. *The product of any* n *successive integers is divisible by* $\lfloor n$.

Let m be the greatest of the integers.

Then $\dfrac{m(m-1)\ldots(m-n+1)}{\lfloor n}=$ the number of combinations of m things taken n together, and is $\therefore$ an integer; $\therefore m(m-1)\ldots(m-n+1)$ is divisible by $\lfloor n$.

Cor. If m is a prime and greater than n, it is prime to each of the numbers 1, 2, 3, . . . n, and therefore to $\lfloor n$;

$\therefore \lfloor n$ must divide $(m-1)\ldots(m-n+1)$, Art. 217;

$$\therefore \frac{(m-1)\ldots(m-n+1)}{\lfloor n}\text{ is an integer,}$$

$$\text{and } \frac{m.(m-1)\ldots(m-n+1)}{\lfloor n}\text{ a multiple of } m.$$

In the same way, if we recognise any expression, in the form of a fraction, to be the sum of a number of integers, we know that its numerator is divisible by its denominator.

EXAMPLES.—XLIX.

1. Show that $n(n+1)(2n+1)$ is always divisible by 6.

2. Show that $(p+2)(p+3)\ \ldots\ (p+n)$ is divisible by $\underline{|n}$, if $p+1$ is prime.

3. If n, p, q be all integers and $n=p+q$, prove that $\dfrac{\underline{|n}}{\underline{|p}\,\underline{|q}}$ is a whole number.

4. If p be a prime, the coefficients of $(1+x)^{p-1}$ will differ from multiples of p by 1, in excess or defect alternately.

5. If p be a prime, and A_0, A_1, A_2, etc. be the coefficients of $(1+x)^{p-2}$, then A_0-1, A_1+2, A_2-3, A_3+4, etc. will be multiples of p.

233. Fermat's Theorem. *If* p *be a prime*, b *a number prime to* p, *then* $b^{p-1}-1$ *is divisible by* p.

$$\text{If}\quad b,\ 2b,\ \ldots\ (p-1)b, \qquad . \qquad . \qquad \text{(I.)},$$

be divided by p, the remainders must be the numbers $1, 2, \ldots (p-1)$.

(1.) The remainders are all different. For, if possible, let two of the numbers in (I.), say mb and nb, give the same remainder r; denote the quotients by s and t;

$$\therefore\ mb=sp+r,$$
$$nb=tp+r;$$
$$\therefore\ (m-n)b=(s-t)p;$$

$\therefore$ p divides $(m-n)b$, but is prime to b, and therefore divides $m-n$; but this is absurd, since m and n are both less than p; $\therefore$ the remainders are not the same.

(2.) No remainder is 0 since p is prime to each factor of each dividend, and $\therefore$ cannot divide any one of the dividends.

(3.) Hence the remainders being all different their number must be $p-1$, and since each must be $<p$ and >0, they must be the numbers 1, 2, . . . $p-1$, though not occurring in this natural order, and the numbers in (I.) must be of the form,

$$n_1p+1,\ n_2p+2,\ \ldots\ n_{p-1}p+p-1;$$

$\therefore$ their product is a multiple of $p+\underline{|p-1}$,

$$i.e.,\ \underline{|p-1}\,b^{p-1}= \text{ a multiple of } p+\underline{|p-1};$$

$$\therefore\ \underline{|p-1}\,(b^{p-1}-1) \text{ is divisible by } p.$$

But p is prime to each factor of the product $\underline{|p-1}$, and $\therefore$ to $\underline{|p-1}$ itself;

$$\therefore\ b^{p-1}-1 \text{ is divisible by } p.$$

234. Fermat's Theorem. Another Proof.

If n *be a prime and* N *prime to* n, *then* $N^{n-1}-1$ *is divisible by* n.

We have

$$(a+b+c+\text{ etc.})^n=a^n+b^n+\text{ etc. }+\frac{\underline{|n}}{\underline{|\alpha}\,\underline{|\beta}\,\underline{|\gamma}\ldots}a^\alpha b^\beta\ \ldots\ +\text{etc.}$$

On the right hand of this equality each term, which is not simply the nth power of one of the symbols a, b, etc., has a coefficient of the form $\dfrac{\underline{|n}}{\underline{|\alpha}\,\underline{|\beta}\ \ldots}$, which, being the number of combinations of n things taken altogether, α being alike, β alike, etc., is a whole number. But n being a prime is not divisible by any factor in $\underline{|\alpha}\,\underline{|\beta}\ \ldots$; $\therefore$ $\dfrac{\underline{|n-1}}{\underline{|\alpha}\,\underline{|\beta}\,\underline{|\gamma}\ \ldots}$ is an integer, and the term is a multiple of n. Hence we may put

$$(a+b+c+\text{ etc.})^n-(a^n+b^n+\text{ etc.})= \text{ a multiple of } n.$$

Now put $a=b=c=$ etc., and let there be N of these symbols,

$$\therefore\ N^n-N,\ i.e.,\ N(N^{n-1}-1)= \text{ a multiple of } n;$$

but, N being prime to n, it is $N^{n-1}-1$ which is divisible by n.

235. *Ex.* 1. If n is >2, it is odd; $\therefore n-1$ is even;

$$\therefore \text{ we may put } N^{n-1}-1=(N^{\frac{n-1}{2}}-1)(N^{\frac{n-1}{2}}+1);$$

$$\therefore N^{\frac{n-1}{2}}-1, \text{ or } N^{\frac{n-1}{2}}+1, \text{ must be divisible by } n;$$

$$\therefore N^{\frac{n-1}{2}} \text{ is of the form } mn\pm 1.$$

Ex. 2. If x is prime to 91, $x^{12}-1$ is divisible by 91.

For since x is prime to 91, it is to 13 ($91=13.7$), and 13 is a prime; $\therefore x^{12}-1$ is divisible by 13.

Also x is prime to 7, and 7 is a prime; $\therefore x^6-1$ is divisible by 7, but $x^{12}-1=(x^6+1)(x^6-1)$, and $\therefore$ is divisible by 7. Now 7 and 13 are prime to one another;

$$\therefore x^{12}-1 \text{ is divisible by their product } 91.$$

EXAMPLES.—L.

1. If N be a prime greater than 2, N^6-1 is divisible by 28.

2. If n be a prime $>N$, then $N^{n-2}+N^{n-3}+\ldots+N+1$ is divisible by n.

3. If p be prime and N prime to p, $N^{1+2+\cdots+p-1}\pm 1$ is divisible by p^2.

4. Prove that $x^{12}-y^{12}$ is divisible by 91, if x and y be prime to 91.

5. If n be a prime, $2^{n-1}+3^{n-1}+\ldots+\overline{n-1}|^{n-1}+2$ is divisible by n.

6. If p be a prime not a sub-multiple of a, then the sum of the remainders, when a, a^2, ... a^{p-2} are divided by p, is less by 1 than a multiple of p.

7. If p be a prime, a and $a-1$ integers not multiples of p, and m any integer, show that $a^{m+1}+a^{m+2}+\ldots+a^{mp}$ is divisible by p.

8. If p be a prime number and n not divisible by p, show that $n^{(p+1)p(p-1)}-n^{p(p-1)}$ is divisible by p^3.

9. If n be a prime number and x any integer, prove that the remainder, on dividing $x^{\frac{n-1}{2}}$ by n, is either 0, 1, or $n-1$.

10. If n be a prime number and N be not divisible by n, prove that $N^{n^r-n^{r-1}}-1$ is divisible by n^r.

236. Wilson's Theorem. *If* p *is a prime,* $1+|\underline{p-1}$ *is divisible by* p.

Let b be one of the numbers $2, 3, \ldots \overline{p-2}$, . (1),
then b is prime to p, and therefore if the numbers

$$b,\ 2b,\ 3b,\ (b-1)b,\ b^2,\ (b+1)b,\ \ldots\ (p-1)b,$$

be divided by p, the remainders are all different (Art. 233, 1); $\therefore$ one and one only is 1.

Also 1 is not the remainder from the division of b, b^2, $b(p-1)$; for b when divided by p gives b for remainder, and $b(p-1)$, or $(b-1)p+p-b$, gives $p-b$ for remainder, and if b^2 gave 1 for remainder, b^2-1, *i.e.* $(b-1)(b+1)$, would be divisible by p, but this cannot be (Art. 217, *Obs.*), since both $b-1$ and $b+1$ are prime to p. Hence one and only one of the products $2b, 3b \ldots b(b-1), b(b+1) \ldots b(p-2)$ give 1 for remainder when divided by p. Hence for each of the numbers in (1) we can find one *other* and only one, such that the product of the pair is of the form $pn+1$, n being an integer.

But the product of any number of integers of this form is still of the same form, viz., $1+$ a multiple of p; $\therefore 2.3.4 \ldots \overline{p-2}$ is of this form, being the product of all such pairs of factors as described above;

$$\therefore 2.3.4 \ldots \overline{p-2}=p.m+1,$$

m being an integer. Multiplying by $p-1$, we have

$$|\underline{p-1}=p.m(p-1)+p-1;$$

$$\therefore |\underline{p-1}+1=p(mp-m+1),$$

and $\therefore$ is divisible by p.

237. *Ex.* I. If p is a prime and >2, $\left\{\underline{\left|\frac{p-1}{2}\right.}\right\}^2+(-1)^{\frac{p-1}{2}}$ is divisible by p.

For p is of the form $2n+1$, where n is an integer;

$\therefore\ 2n=p-1,\ 2n-1=p-2, \ldots\ n+2=p-\overline{n-1},\ n+1=p-n;$

$\therefore\ 2n(2n-1)\ \ldots\ (n+2)(n+1)$ is a multiple of $p+(-1)^n\underline{|n}$;

$\therefore\ 1+\underline{|p-1}=1+\underline{|2n}=1+\underline{|n}.(n+1)(n+2)\ \ldots\ 2n$

$=1+$ a multiple of $p+(-1)^n\{\underline{|n}\}^2$;

$\therefore\ 1+(-1)^n.\{\underline{|n}\}^2$, and $\therefore\ (-1)^n+\{\underline{|n}\}^2$, is divisible by p.

If p is of the form $4m+1$, $n=2m$; $\therefore\ (-1)^n=1$, and p is a factor of the sum of two squares, 1 and $\{\underline{|n}\}^2$.

EXAMPLES.—LI.

If n be a prime, prove that

1. $2(n-1)\underline{|n-3}-1$ is divisible by n.

2. $\dfrac{1^2.2^2.3^2.4.5\ \ldots\ (n-5)(n-4)(n-1)+1}{n}$ is an integer.

3. $1-(-1)^p\underline{|p-1}\,\underline{|n-p}$ is divisible by n.

4. The series of squares $1^2, 2^2, 3^2, \ldots \left(\frac{a-1}{2}\right)^2$ leave each a different positive remainder when divided by a, if a is a prime.

238. PROP. *To find the highest power of a prime* a, *which is contained in the product* $\underline{|m}$.

Let N_1 be the number of integers not $>m$ containing a but not a^2,

N_2 „ „ a^2 „ a^3,

and so on.

Hence amongst the numbers $1, 2, 3, \ldots m$ we have a repeated N_1 times; $\therefore\ a^{N_1}$ is a factor of $\underline{|m}$.

Also we have a^2 repeated N_2 times; $\therefore\ a^{2N_2}$ is a factor of $\underline{|m}$,

and „ a^3 „ a^{3N_3} „

and so on; $\therefore\ a^{N_1+2N_2+3N_3+\ldots}$ is a factor of $\underline{|m}$,

and this is the highest power of a in $\underline{|m}$.

We will now express this in a form more convenient for application.

Let n_1 denote the greatest integer in $\frac{m}{a}$,

n_2 " " $\frac{m}{a^2}$,

n_3 " " $\frac{m}{a^3}$, etc.

Then the integers not $>m$, containing a, are

$$a,\ 2a,\ 3a\ \ldots\ n_1 a;$$

$\therefore$ there are n_1 integers which contain a, including those which contain a^2, a^3 . . . ;

$$\therefore\ n_1 = N_1 + N_2 + N_3 + \text{ etc.}$$

Similarly there are n_2 integers which contain a^2, a^3, etc.,

" n_3 " " a^3, a^4, "

$$\therefore\ n_2 = N_2 + N_3 + \text{ etc.}$$
$$n_3 = N_3 + N_4 + \text{ etc.};$$
$$\therefore\ n_1 + n_2 + n_3 + \text{ etc.} = N_1 + 2N_2 + 3N_3 + \text{ etc.}$$

239. Induction is often useful in problems where it is required to prove that a certain form is divisible by a given number.

Ex. 1. The integral part of $(3+\sqrt{5})^n+1$ is divisible by 2^n, n being a positive integer.

Let I denote the integral part of $(3+\sqrt{5})^n$, F its fractional part;

$$\therefore\ I+F=(3+\sqrt{5})^n.$$

Now $3-\sqrt{5}$, and $\therefore\ (3-\sqrt{5})^n$, is a proper fraction.

Put $F'=(3-\sqrt{5})^n$.

Therefore $I+F+F'=(3+\sqrt{5})^n+(3-\sqrt{5})^n$

$=$ an integer, since all the irrational parts disappear, and all the coefficients are integers (Art. 232);

$$\therefore\ F+F' \text{ is an integer, and } \therefore\ =1 \text{ (see Art. 103)};$$
$$\therefore\ I+1=(3+\sqrt{5})^n+(3-\sqrt{5})^n=S_n \text{ say.}$$

When $n=1$, we have $S_1=6=2.3$,

$n=2$, " $S_2=2(3^2+5)=28=2^2.7$;

$\therefore$ the theorem is true when $n=1$, and $n=2$.

Suppose it to be true, when $n=m$, and when $n=m+1$, say $S_n=2^m.a$, $S_{n+1}=2^{m+1}b$.

Now $S_{m+1}=(3+\sqrt{5})^{m+1}+(3-\sqrt{5})^{m+1}$,

and $6=(3+\sqrt{5})+(3-\sqrt{5})$;

$$\therefore 6S_{m+1}=(3+\sqrt{5})^{m+2}+(3-\sqrt{5})^{m+2}$$
$$+(3+\sqrt{5})(3-\sqrt{5})\{(3+\sqrt{5})^m+(3-\sqrt{5})^m\},$$
$$=S_{m+2}+4S_m;$$
$$\therefore S_{m+2}=6.2^{m+1}b-4.2^m.a=2^{m+2}(3b-a);$$

$\therefore$, by Induction, S_{m+2} is divisible by 2^{m+2}.

240. *Ex.* 2. To prove that $2^{2n}-3n-1$ is divisible by 9.

Let $f(n)=2^{2n}-3n-1$;

$$\therefore f(n+1)=2^{2n+2}-3(n+1)-1;$$
$$\therefore f(n+1)-f(n)=2^{2n+2}-2^{2n}-3=3.2^{2n}-3=3(2^{2n}-1)$$
$$=3(2^n-1)(2^n+1).$$

Now

$$2^n=(3-1)^n=3^n-n.3^{n-1}+\frac{n(n-1)}{1.2}\cdot 3^{n-2}-\text{etc.}\pm n.3\mp 1$$
$$=\text{ a multiple of } 3\mp 1;$$

$\therefore$ either 2^n+1, or 2^n-1, is always divisible by 3;

$\therefore f(n+1)-f(n)$ is divisible by 9;

$\therefore$ when $f(n)$ is divisible by 9, $f(n+1)$ is also.

Now $f(2)=16-6-1=9$;

$\therefore f(3)$ is divisible by 9; $\therefore f(4)$ is also, and so on generally.

EXAMPLES.—LII.

1. If n be any prime number except 2, the integral part of $(\sqrt{5}+2)^n-2^{n+1}$ is divisible by $20n$.

2. Show that, if x be any prime number except 2, the integral part of $(1+\sqrt{2})^x$ diminished by 2 is divisible by $4x$.

3. Show that $2^{2n+1}-9n^2+3n-2$ is a multiple of 54.

4. Prove that $7^{2n}+16n-1$ is divisible by 64, if n be a positive integer.

5. Find the highest powers of 5 and of 25 contained in $\lfloor 30$.

6. " " 3 " 6 " $\lfloor 16$.

241. Prop. *The number of primes is infinite.*

For let p be a prime; then the continued product $1.2.3.5.7 \ldots p$ of all the primes up to p is divisible by each of them; $\therefore 1.2.3.5 \ldots p+1$ is not divisible by any one of them except 1.

Now let $1.2 \ldots p+1$ be resolved into its prime factors. Since it is not divisible by any number so small as p, each of its prime factors must be greater than p, *i.e.*, there *is* a prime $>p$.

Thus we have shown that whatever prime p may be, there is always a prime greater than it. Hence the number of primes is infinite.

242. Any prime >3 is of the form $6m \pm 1$, Art. 227, *Ex.* 3.

But every number of the form $6m \pm 1$ is not necessarily a prime; and in fact

No Algebraical formula can represent primes only.

For let $a+bx+cx^2+$ etc. be an Algebraical expression which represents primes for some values of x, the symbols a, b, c, etc. representing constant numbers.

Let $a+bx+cx^2+$ etc. $=$ a prime p, when $x=m$.

Put $x=m+np$, then the expression becomes

$$a+b(m+np)+c(m+np)^2+\text{ etc.}$$
$$=a+bm+cm^2+\text{ etc. }+\text{ a multiple of } p$$
$$=p+\text{ a multiple of } p.$$

Hence the number now represented by the expression has p for a factor, and therefore is not a prime.

Therefore the expression $a+bx+cx^2+$ etc. does not always represent a prime for every value of x.

For further information on the subject of this Chapter, see Peacock's *Algebra*, Barlow's *Theory of Numbers*, Serret's *Algèbre Supérieure*.

XXI

Probabilities.

243. In the problems we are about to discuss the chief difficulty is in understanding the ideas involved, and the terms employed; the Mathematics necessary for their solution being of a very simple character.

We shall therefore venture, by way of introduction, upon a somewhat lengthy explanation, for which we beg the student's careful attention.

244. We commence with the following example.

Ex. A record has often been kept of the ages of all persons who have died at a certain place through a long sequence of years.

Now, on examining such a record up to the end of each year, suppose we find the numbers, amongst those who have died up to that instant *since the beginning of the record*, (1) of those who were over 1 year of age at the time of death, (2) of those who were over 26. These numbers would continually increase, of course, as we continued to take in, at each examination, one more year, and we should see that the ratios of the *second* to the *first* differed continually less and less from some one ratio.

As a matter of fact it was found at one place that this ratio was 560 : 1000, or 14 : 25.

We should then say that the *probability* of any person, whose age at death had been recorded, and whom we knew to be over 1 year at death, being also over 26 years, was 14 : 25.

Here we considered a succession of persons, each possessing one *general* characteristic, namely, dying at this place after reaching the age of 1 year. This characteristic is common to them all, and distinguishes them from all others whose deaths are recorded, so that by it we are able to class them together.

Further in this succession we considered a certain *particular section*, each member of which, besides the general characteristic of being 1 year old, or over, at death, possessed an additional characteristic of being 26 years, or over, at death; and we sought the probability of any one, whom we are told belonged to the *general* class, belonging also to the *particular* section.

245. The following is a general *definition* of the term *Probability*.

Suppose that we have a succession of things (or events or persons), each possessing some general characteristics, on account of which they can be classed together, and that amongst them is a section, each member of which, besides the general characteristics, possesses an additional attribute, which distinguishes this particular section from all the other things. Then if, on taking any *large* number of the general things, we find that the number of particular things amongst them tends to bear to it a constant ratio, this ratio is called the *Probability of any one of the general being also one of the particular things.*

246. *Ex.* Thus if, by taking notice of a large number of ships, of the same class, and sailing under the same circumstances, we find that *on the average* three in a hundred are wrecked, we say that the probability of any one of such ships being wrecked is $3:100$.

247. Of course this ratio, like all others, is represented by a fraction, having the antecedent and consequent as numerator and denominator. Thus in Art. 246 the algebraic representative of the probability is $\frac{3}{100}$.

248. In most of the cases, with which we have to deal in practice, it is only when we take *large numbers* of the general succession that we find that the ratio is constant. At first, when small numbers are taken, the ratios have large differences, but if by taking more and more of the succession we find that the

fractions differ by less and less from some one fraction, then we say that this is their limit and represents the probability considered. If no such limit exists, then there is no definite probability that any one of the general succession is also one of the particular class.

The reader must bear in mind then, that it is only when large numbers are considered that the results of our investigations can be expected to correspond, with any degree of accuracy, to the actual state of things. Thus we should make our calculations as if 3 wrecks had occurred in every 100 ships that had sailed, whereas actually a wreck might not have occurred till we came to the 500th sailing. Also in Art. 244 we should reason as if 14 out of every 25, who were more than one year of age, survived to the age 26; but it would be only when we worked with large numbers that our results would agree with experience.

249. *Ex.* Again, suppose that a ball is drawn from a bag and then put back, and this repeated for a great number of trials, and that we draw white, black, and red balls, and no others. If now, on keeping records of the number of times a white, black, and red ball is drawn, it is found that the ratios of the number of appearances of the white, black, and red balls to the whole number of drawings tended respectively to the following ratios:—27 : 100, 53 : 100, 20 : 100, then the probabilities that any one particular drawing had produced a white, black, or red ball would be represented by $\frac{27}{100}$, $\frac{53}{100}$, $\frac{1}{5}$.

Also it would be said that it was 27 to 73 for, and 73 to 27 against, a white ball having appeared at any one drawing; and that the odds were 80 to 20 (*i.e.*, 4 to 1) in favour of, and 1 to 4 against, a white *or* a black ball having appeared at any one drawing.

250. We have supposed that notice has been taken of how many only, and not which, of the general succession belonged also to the particular section. That is to say, we are ignorant

of the particular circumstances of each individual thing, having a knowledge of the aggregate only. This is sufficient to enable us to settle the probability required.

251. If every one of the general events belongs to the particular section, then it is said to be a *certainty* that any one of the general events belongs to the particular section. Also the number of those belonging to the particular section is equal to the number of general events. Hence the probability of a certainty is represented by 1.

252. The words *chance* and *probability* are used as synonymous.

253. The following is an instance of the use we can make of probabilities.

In many cases, when we have actually observed only a certain portion of a succession, we are enabled to infer, by induction, that the probability, found to exist amongst the observed members of the succession, exists also amongst those which have not been observed.

Thus we should conclude that $3:100$ is the probability of any ship being wrecked, of the same class, and sailing under the same circumstances as those in Art. 246; taking no account of whether it is a ship that has sailed in the past or will sail in the future; and that $\frac{14}{25}$ represents the probability of *any* infant of the age of 1 year living to be 26, if the circumstances of his life do not differ from the generality of those of which a record was made.

254. We add some more examples.

Ex. 1. If, on the average 1 man in 10 is under 5 ft. 6 in., and 3 in 40 have red hair, what is the probability that any one man is both red-haired and under 5 ft. 6 in., it being assumed that the colour of a man's hair has no effect on his stature?

Since in every 40 men 3 have red hair; ∴ in every 400 men 30 have red hair.

Now 1 in every 10 is under 5 feet 6 inches; ∴ amongst the 30 who have red hair there are 3 who are under 5 feet 6 inches. Hence in every 400 men, 3 are both red-haired and under 5 feet 6 inches, *i.e.*, the probability required is 3 : 400.

Ex. 2. If one man in 30 is 6 feet at least, and, of those under 6 feet, 1 in 40 is 5 feet or under, what is the probability of a man being between 5 and 6 feet?

Since 1 man in 30 is 6 feet at least, 29 in 30 are under 6 feet.

Now consider 1200 men. Of these 1160 are under 6 feet; and, since 39 in 40 are over 5 feet, ∴ of the 1160 men 1131 are over 5 feet.

Hence of 1200 men, 1131 are within the specified limits, or the probability required is 1131 : 1200, or 377 : 400.

Ex. 3. If 1 man in 30 is 6 feet or over, and 1 in 40 is 5 feet or under, what is the chance of a man being between 5 and 6 feet?

Out of 1200 men, 40 are over 6 feet, and 30 are 5 feet at most; ∴ 1200−70, or 1130, are between 5 and 6 feet; ∴ the required chance is 113 : 120.

255. In a large number of cases we have no observations, on any portion of the succession, recorded, but are expected to determine what the aggregate of the succession will be from *à priori* considerations, drawn from the circumstances under which it is to take place.

Thus, if we have 3 white, 4 black, and 11 red balls in a bag, we determine what the aggregate will be from the assumption, that, in the long run, out of every 18 times that a ball is drawn and put back, a white ball will be drawn 3, a black 4, and a red 11 times.

Again, if a coin is tossed, it may be fairly assumed that it will turn up heads as often as tails.

Conversely, if we knew that we had 12 balls in a bag, and that the chance of drawing a white ball was $\frac{1}{3}$, we should conclude that there were 4 white balls.

256. And in general, if there are b ways in which an event may happen, and there is no *à priori* reason why one should occur rather than another (or, as it is sometimes expressed, all the ways are equally likely), and if a is the number of ways in which the event may occur, so as to be of a particular section, then it is assumed that, in the long run, it will be of this section a out of every b times that it happens. Hence the probability of its thus happening is $a:b$.

This is the same as assuming that this known numerical ratio, existing amongst the *possible*, or *equally likely*, ways, is a cause so much more efficient than any other for determining the aggregate of the succession, that the effects of all other causes may be neglected in comparison.

EXAMPLES.—LIII.

1. What is the chance of an ace being thrown with one die? What is the chance of an ace *or* a six being thrown?

2. What is the chance of a six being drawn from an ordinary pack of cards?

3. Show that it is 10 to 3 against a court card being drawn from a pack.

4. What are the odds against drawing a black or a white ball from a bag containing 5 white, 6 black, and 7 red balls?

5. What is the chance of drawing, from an ordinary pack of cards, a court card of the club suit?

6. A bag contains 20 sovereigns and 30 shillings, of which 3 in every 10, whether sovereigns or shillings, were coined in the present reign. What is the chance of drawing a Victorian sovereign, supposing that I cannot tell a sovereign from a shilling by the touch?

7. Two bags contain the same number of balls, the chance of drawing a red ball from one is $\frac{3}{10}$, and from the other is $\frac{4}{15}$. If all the balls are put into one bag, required the chance of drawing a red ball from it.

8. A bag contains white, black, and red balls, 12 in all. The chances of drawing the various kinds are respectively $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{6}$. Find the number of each kind.

9. A bag contains white and black balls only, and the odds against a white ball being drawn are 3 to 2. What is the chance of a white ball being drawn? What is the chance for a black?

10. One of two events must happen. The odds against one event are 7 : 5. What is its chance of happening?

11. The chance of drawing a white ball from a bag is $\frac{5}{6}$. Are the odds for or against such a ball being drawn, and what are they?

257. In working out such problems as are referred to in Art. 256, the difficulty usually consists in determining, (1) the number (b) of different ways in which the general event may happen, and (2) the number (a) of these ways in which it may happen so as to belong to the particular section.

258. *Ex.* 1. What is the chance of throwing exactly 12 with 3 dice?

Here the general event is the throwing 3 dice, and the number of different ways in which this may be done is $6\times6\times6=216$; and, in order that any one of these may belong to the particular section, the sum of the numbers turned up must be 12. It will be seen that the following 25 throws are those which give 12:—

651, 615, 561, 165, 516, 156,
642, 624, 462, 264, 426, 246,
633, 363, 336, 552, 525, 255,
543, 534, 453, 354, 435, 345, 444.

Hence the chance of throwing 12 is $\frac{25}{216}$.

Ex. 2. Find the probability of an ace being thrown, once at least, in two throws with a die.

To make any one of the double throws, any one of the faces

at one throw may be followed by any one of the faces at the second throw, thus we have 36 possible different double throws.

Now an ace may occur in any one of 11 double throws, namely, an ace may be followed by an ace or by any one of the other 5 faces, or an ace at the second throw may be preceded by any one of the 5 other faces at the first throw, and, there being no *general* reason why one double throw should occur more often than another, we assume that an ace will occur, at least once, 11 times out of 36, *i.e.* the probability of its occurring is $\frac{11}{36}$.

The chance that an ace occurs at both throws is $\frac{1}{36}$.

The chance that an ace occurs at the second throw, and *not* at the first, is $\frac{5}{36}$; and that it occurs once, but either at the first or at the second throw, is $\frac{10}{36}=\frac{5}{18}$.

Ex. 3. What is the chance of two white balls being drawn in succession from a bag containing 5 white and 6 black; the first ball drawn not being put back?

Here the general event is the drawing two balls; and, since any one ball may be followed by each of the others, the number of different ways of drawing two balls is $11\times10=110$.

The drawings in which two whites are produced form the particular section, and the number of these is $5\times4=20$. For to produce such a drawing any one of the 5 whites must be followed by one of the remaining 4.

Hence the chance required $=\frac{2}{11}$.

So the chance of a white being followed by a black is $\frac{30}{110}=\frac{3}{11}$,

" black " white $\frac{3}{11}$.

" white and black being drawn in either order $=\frac{6}{11}$.

Ex. 4. One of a pack of 52 cards having been removed, from the remainder of the pack two cards are drawn and found to be spades. Find the chance that the missing card is a spade.

The missing card may be, and is equally likely to be, any one of the pack except the two drawn, *i.e.*, any one of 50 cards. Also there are 11 cards, any one of which it may be so as to be a spade. Hence the probability of its being a spade is $\frac{11}{50}$.

EXAMPLES.—LIV.

1. A ball is drawn from a bag containing 5 white and 6 black, and replaced, and then another drawing made. Find the chances (1) that both balls are white, (2) that a black ball is followed by a white, (3) that a black and a white ball are drawn.

2. Compare the chance of throwing 10 with that of throwing 7 with a common pair of dice.

3. A throws with a common die, B with one in which 5 is counted as a blank. What is the chance that A throws higher than B?

4. A and B play with 3 dice, and the highest wins. A throws 14; what are the chances of B's winning?

5. There are 3 balls in a bag, and each of them may with equal probability be white, black, or red. A person puts in his hand and draws a ball; it is white; it is then replaced. Find the chance of all the balls being white.

6. A bag contains 5 white and 6 black balls. What is the chance of drawing two white balls together? What is the chance that a black and a white ball are drawn together?

7. In 3 throws with a single die, find the chances that an ace is thrown (1) at least once, (2) only once, (3) at least twice, (4) only twice.

8. What are the chances of throwing (1) 8, (2) 10, with two dice?

9. Out of a heap of 10 counters, numbered from 1 to 10, a counter is drawn and replaced 4 times. What is the chance that the sum of the numbers drawn is 33?

10. What are the chances of throwing with 3 dice at one throw, (1) 10, (2) not more than 10?

11. In three throws with a pair of dice, what is the probability of having doublets once at least?

12. A man is known to have in his pocket half-a-crown in small silver. A coin taken from it at random is found to be a shilling; show that the chance of his having another shilling is $\frac{1}{4}$.

13. A bag contains 6 coins, each either a shilling or a sovereign; the *à priori* probability that any particular coin is a sovereign is $\frac{1}{3}$. Supposing two shillings and one sovereign have been drawn out of the bag, what would you offer for the remaining three coins?

14. If a thousand counters be numbered from 1 to 1000, find the chance of drawing a number prime to 1000.

15. When a coin is tossed, what is the chance that it falls heads twice running?

16. In a bag are 3 white, 4 black, and 5 red balls. If two balls are drawn, what are the chances that they are (1) one white and the other black, (2) both red, (3) one at least red?

17. In one throw with a pair of dice, what is the chance that there is neither a six nor doublets?

259. The results of Chapters [XXXIII., XXXIV.], will often be found useful in determining a and b (Art. 256). In fact, solutions of chance problems of this class merely consist of two or more applications of the rules for Permutations and Combinations.

Ex. Seven gentlemen and six ladies meet at a croquet party, at which the game consists of two ladies and two gentlemen on each side; find the chance of a given couple playing on the same side.

1° For the general event.

The number of different quartettes of gentlemen possible $=\frac{7.6.5.4}{1.2.3.4}=35.$

The number of different quartettes of ladies possible $=\frac{6.5.4.3}{1.2.3.4}=15.$

Now, taking any one quartette of gentlemen playing with any one quartette of ladies, we can form 6 pairs of gentlemen and 6 pairs of ladies, and therefore 36 different sides; but we must have two sides in each game. Hence we have 18 games which

can be formed by the same set of 8; $\therefore$ in all we have 35.15.18, or $7.5^2.3^3.2$, different games.

2° For the particular section.

There are $\frac{6.5.4}{1.2.3}$, or 20, different quartettes of gentlemen, in which any one gentleman plays, and 10 different quartettes of ladies, in which any one lady plays, and, hence, 200 different arrangements, so that one couple play in the same game.

Now they are as often opposed as together. Hence the number of games in which they are on the same side is $100 = 5^2.2^2$;

$$\therefore \text{ the chance required } = \frac{2}{7.3^3} = \frac{2}{189}.$$

EXAMPLES.—LV.

1. Four white and 3 black balls are placed in a line; what is the chance that the end balls are white? What are the odds against both end balls being black?

2. What is the chance of drawing 2 black balls and 1 white from an urn containing 5 white, 4 black, and 2 red balls?

3. A person draws 3 coins from a bag containing 4 sovereigns and 4 shillings; what are the odds against them being all sovereigns?

4. From a bag containing 1 white, 2 red, and 3 black balls, two are drawn; what are the chances that they are both (1) red, (2) black?

5. If 20 cards are drawn from a pack, what are the chances that they contain (1) the 13 clubs, (2) the 4 kings, (3) the 13 clubs *and* the 4 kings?

6. Eight persons are to play at croquet. Four of them wish to play on the same side, but take up four mallets at random. The sides are decided by the order of the colours on the stick. Find the probability that these four will play together.

7. A, B, C, D, E take a four oar, determining each man's place by lot, find the chance that A and B are the stroke side oars.

8. What is the chance that a person who deals at whist will have four trump cards and no more?

9. There are n books on a shelf which are taken down, dusted and returned. If all the ways of arranging them with their backs outwards are equally likely to be made in returning them, find the chance that they are replaced exactly as they were, (1) when they are all put right ends upwards, (2) when right ends and wrong ends upwards are equally likely.

10. A thief in a dark night catches four birds in an aviary, which contained altogether five canaries, four goldfinches, three nightingales, and two robins. What is the probability of his catching one of each kind?

11. Eight people sit down to a round table. Show that the number of ways in which they can be arranged is to the number of arrangements so that the same two shall sit together as 7 : 2.

12. Ten people sit down to a round table. Show that it is 7 to 2 against any two given people sitting next one another.

13. What is the probability of drawing 4 aces from a pack in 4 successive trials?

14. If the House of Commons consist of m Tories and n Whigs, and a committee of $p+q$ members be selected by ballot, what is the chance that it contain p Tories and q Whigs?

15. In 5 throws with a single die, find the chance that an ace will be thrown at least twice.

16. An urn contains 3 white, 4 red, and 5 black balls; what are the chances of drawing (1) 1 white and 1 red and 1 black in 3 successive trials; (2) 2 white and 2 red in 4 successive trials; (3) 1 white, 2 red, 3 black in 6 trials?

17. At a game of whist, what is the chance of dealing one ace and no more, (1) to a specified person, (2) to each person?

18. In a lottery there are 100 numbers, of which 5 are drawn, what is the chance that 3, and only 3, out of 5 previously specified numbers, are drawn?

19. If we draw 4 cards out of a pack, what is the chance that they each belong to a different suit?

20. An urn contains 20 balls, namely, 6 white, 4 black, 3 red, and 7 blue, what is the chance that, in 9 drawn at one time, we shall have 2 white, 3 black, 1 white, and 3 blue?

260. We shall now pass on to some more general theorems relating to probabilities.

The word *event* is applied not only to the general thing, as in Art. 245, but sometimes also to any one of the particular ways, in which the general event can happen.

Thus, in Art. 255, the drawing of each ball is an event; but also we talk of the event of the ball when drawn being red, white, or black.

Single Exclusive Events.

261. *Def.* Events are called *exclusive*, if, when one of them does, no other one can, happen.

Thus the drawing of single white, black, or red balls, in Art. 255, would be exclusive events.

262. *If* p *and* q *be the respective probabilities of two exclusive events* (call them A and B) *happening on any one of a succession of occasions, when one or other must happen, then* $p+q=1$.

Let $p=\frac{a}{b}$. Out of the b occasions when A *may* happen, we have, a on which A *does* happen, and $\therefore$ on which B does not happen,

$b-a$ " " not " " " happen;

$$\therefore q=\frac{b-a}{b}; \therefore p+q=1.$$

Cor. If a succession of trials be made to do anything, which must either succeed or fail, then if p be the chance of any one trial succeeding, $1-p$ is the chance of its failing.

If the chance of an event happening is $\frac{1}{2}$, the chance of its not happening is $\frac{1}{2}$; and the chances of its happening, and not happening, are said to be *even;* or, it is said to be an even chance whether it happens or not.

263. Prop. *If there be* n *exclusive events*, $A_1, A_2, \ldots A_n$, *and* $p_1, p_2, \ldots p_n$ *be the chances of their respectively happening on any one of a succession of occasions, on which one must happen, then the chance that, on any given occasion, one of the first* r *events happens is* $p_1+p_2+\ldots+p_r$.

Let $p_1=\frac{a_1}{c}$, $p_2=\frac{a_2}{c}, \ldots p_n=\frac{a_n}{c}$; so that, out of every c occasions, A_1 happens on a_1, and A_2 on a_2, and so on.

Since no two can happen on the same occasion, therefore on $a_1+a_2+\ldots+a_r$, out of c, occasions one or other of the first r events happens; $\therefore$ the chance of some one of them happening on any given occasion is $\frac{a_1+a_2+\ldots+a_r}{c}=p_1+p_2+\ldots+p_r$.

Cor. If p be the chance of any one of r exclusive events, all equally likely to happen, then rp is the chance of some one happening.

Compound Events, Components being Exclusive.

264. The following example will enable the student to understand more easily the method pursued in Art. 265.

The chance that a white ball is drawn from a bag is $\frac{3}{10}$, and $\frac{7}{10}$ that a black is drawn. Find the chance that, in any sequence of 3 drawings, 2 white and 1 black are drawn, each ball being put back after it is drawn.

Consider 1000 sequences of 3 drawings each.

In the 1000 drawings which begin these sequences we may calculate that a white ball is drawn 300 times; *i.e.* there are 300 of the sequences which begin with a white ball.

Then in the 300 drawings which come second in these sequences, a white will be drawn 90 times; *i.e.* there are 90 sequences having a white ball at each of the two first drawings.

Then in the 90 drawings at the end of these sequences a black will be drawn 63 times; *i.e.* there are 63 sequences in which 2 white are drawn first and then 1 black. Similarly, 63 is the

number of sequences in which 2 white and 1 black occur in any given order; but we have $\frac{\lfloor 3}{\lfloor 2}$ $(=3)$ different orders of arrangement of 2 white and 1 black balls; $\therefore$ there are 3.63 $(=189)$ sequences out of 1000, in which 2 white and 1 black occur; $\therefore \frac{189}{1000}$ is the chance required.

265. PROP. *A and B are two exclusive events,* p *is the chance of A, and* q *of B, happening on any one of a succession of occasions when either* A *or* B *must happen. To find the chance that, in any given sequence of* n *occasions, A will happen* r *times and B* n—r *times.*

Since $p+q=1$, Art. 262, put $p=\frac{a}{c}, q=\frac{b}{c}$, where $a+b=c$.

Consider c^n sequences of n occasions each.

Of the c^n occasions coming *first* in these sequences, A will occur on ac^{n-1}; so that there are ac^{n-1} sequences in which A occurs on the first occasion.

Then of the ac^{n-1} occasions coming *second* in these sequences, A will occur on a^2c^{n-2}; so that there are a^2c^{n-2} sequences in which A occurs on the first two occasions.

Similarly there are a^rc^{n-r} sequences in which A occurs on the first r occasions.

Of the a^rc^{n-r} occasions coming in the $(r+1)$th place in these sequences, B will occur on a^rbc^{n-r-1}; so that there are a^rbc^{n-r-1} sequences, in which, A occurs on the first r occasions, and then B on the next occasion, and so on generally. Hence we have a^rb^{n-r} sequences, in which, A occurs on the first r occasions, and then B on the remaining $n-r$.

And in the same way we can show that there is the same number of sequences, in which A and B occur, r and $n-r$ times respectively, in any given order of arrangement.

But there are $\dfrac{\lfloor n}{\lfloor r \lfloor n-r}$ different arrangements of n things, in which r are alike and $n-r$ alike.

Hence out of c^n sequences we have $\dfrac{\lfloor n}{\lfloor r \lfloor n-r} a^r b^{n-r}$, in which A occurs r times and B $n-r$ times *in some order or other;* ∴ the chance, that, in any given sequence of n occasions, A and B so occur, is

$$\frac{\lfloor n}{\lfloor r \lfloor n-r} \frac{a^r b^{n-r}}{c^n} = \frac{\lfloor n}{\lfloor r \lfloor n-r} p^r q^{n-r}.$$

Putting successively $r=n, n-1, \ldots 1, 0$, we see that the chance, in any sequence,

that A occurs n times and B not at all, is p^n;

that A occurs $n-1$ times and B once, is $np^{n-1}q$;

that A occurs $n-2$ times and B twice, is $\dfrac{n(n-1)}{1.2} p^{n-2}q^2$;

etc. etc. etc.; and

that A occurs not at all and B n times, is q^n.

COR. Out of c^n sequences the number, in which A occurs at least r times, is

$$a^n + na^{n-1}b + \frac{n(n-1)}{1.2} a^{n-2}b^2 + \ldots + \frac{\lfloor n}{\lfloor r \lfloor n-r} a^r b^{n-r}.$$

Hence the chance, that, in any sequence of n occasions, A will occur at least r times, is

$$p^n + np^{n-1}q + \frac{n(n-1)}{1.2} p^{n-2}q^2 + \ldots + \frac{\lfloor n}{\lfloor r \lfloor n-r} p^r q^{n-r}.$$

266. The student will notice that the chances of the various combinations of the occurrences of A and B are the terms of the expansion of $(p+q)^n$.

Hence the most likely combination is that for which the corresponding term of the expansion of $(p+q)^n$ is greatest.

267. In the same way it may be shown, that the various terms of the expansion of $(p+q+r)^n$ will furnish the chances of the various combinations of the occurrences of three exclusive events, of which the separate chances at any one opportunity are p, q, r.

268. *Ex.* 1. What is the chance of at least one ace being thrown at every one of 3 throws with two dice?

The chance of at least one ace at any one throw is $\frac{11}{36}$.

The chance of at least one ace at every one of 3 throws is $\left(\frac{11}{36}\right)^3$.

Ex. 2. What is the chance of two aces at least being thrown in 3 throws with two dice?

This is the same as throwing 6 times with one die, and then, at any one throw, the chance of one ace is $\frac{1}{6}$, and of no ace is $\frac{5}{6}$;

$\therefore$ the chance of an ace being thrown at least twice is

$$\left(\frac{1}{6}\right)^6+6.\left(\frac{1}{6}\right)^5\cdot\left(\frac{5}{6}\right)+\frac{6.5}{1.2}\cdot\left(\frac{1}{6}\right)^4\left(\frac{5}{6}\right)^2+\frac{6.5.4}{1.2.3}\cdot\left(\frac{1}{6}\right)^3\cdot\left(\frac{5}{6}\right)^3$$
$$+\frac{6.5}{1.2}\cdot\left(\frac{1}{6}\right)^2\cdot\left(\frac{5}{6}\right)^4=\left(\frac{1}{6}\right)^6\{1+6.5+15.5^2+20.5^3+15.5^4\}.$$

We might have shortened the process thus:

The chance that no ace is thrown in 6 times is $\left(\frac{5}{6}\right)^6$,

" only one ace " $6.\left(\frac{5}{6}\right)^5\cdot\frac{1}{6}$;

$\therefore$ the chance that 2 *are not thrown* is $\frac{5^6+6.5^5}{6^6}=\frac{11.5^5}{6^6}$, Art. 263;

$\therefore$ the chance that 2 aces are thrown is $1-\frac{11.5^5}{6^6}$, Art. 262.

Ex. 3. How many throws with a single die must a man have, in order that his chance of throwing an ace may be $\frac{1}{2}$?

Let x be the required number of throws.

His chance of not throwing an ace at one throw is $\frac{5}{6}$;

$\therefore$ " " at all in the x throws is $\left(\frac{5}{6}\right)^x$.

But his throwing an ace once at least and throwing no ace at all are two exclusive events, one of which must happen;

$\therefore$, Art. 262, his chance of throwing an ace is $1-\overline{\frac{5}{6}}\Big|^x$;

$$\therefore 1-\overline{\frac{5}{6}}\Big|^x=\frac{1}{2}; \qquad \therefore \overline{\frac{5}{6}}\Big|^x=\frac{1}{2};$$

$$\therefore x(\log 5-\log 6)=-\log 2; \quad \therefore, \text{ by tables, } x=3.8.$$

This shows that there is no exact number of throws in which his chance will amount to $\frac{1}{2}$, but, if he throws 4 times, his chance will be a little greater than $\frac{1}{2}$.

Ex. 4. If a bag contain 3 white and 7 black balls, and a ball be drawn and always replaced, show that the most likely result in 10 trials is to draw 3 white and 7 black balls.

The chance of a white occurring at any one drawing is $\frac{3}{10}$,

" black " " $\frac{7}{10}$.

Hence we have to find the greatest term in the expansion of $(3+7)^{10}$, Art. 266.

Now [Art. 421] the $(r+1)$th is greatest when r is the integral part of $\frac{(10+1)7}{3+7}$, *i.e.* of $\frac{77}{10}$; $\therefore r=7$.

Hence the eighth term is the greatest, and to this term corresponds the chance of a combination in which 7 black and 3 white balls occur.

Ex. 5. If A's skill at a game is double that of B, what is the chance that he wins 4 games before B wins 2?

By A's skill is meant his chance of winning any single game.

Hence A's skill is represented by $\frac{2}{3}$, B's by $\frac{1}{3}$.

A's chance of winning the *first* four games is $\left(\frac{2}{3}\right)^4$, and his chance of winning 3 out of the first four and the fifth is $4.\left(\frac{2}{3}\right)^3.\frac{1}{3}.\frac{2}{3}=\frac{4}{3}.\left(\frac{2}{3}\right)^4$.

Also these are the only two ways, in which A can win 4 before B wins two, and they are exclusive. Hence, Art. 263, the required chance $=\left(\frac{2}{3}\right)^4\left(1+\frac{4}{3}\right)=\frac{16\times 7}{3^5}=\frac{112}{243}$.

EXAMPLES.—LVI.

1. What are the odds against throwing heads n times in succession in the game of heads and tails?

2. What is the chance of throwing at least four aces with a single die in six throws?

3. When 3 coins are tossed up, what are the chances that, (1) two only turn up tail, (2) two at least turn up tail?

4. A bag contains 5 black and 4 white balls; a person draws a ball and replaces it; what is the chance that in 6 trials he will have drawn black at least 4 times?

5. If on an average 1 ewe out of 3 yields 2 lambs, find the chance that 3 ewes bearing young will yield just 4 lambs.

6. In six throws with a single die, what are the chances of

throwing a six (1) four times exactly, (2) not more than four times, (3) not less than four times?

7. What are the odds against throwing 7 twice at least in 3 throws with 2 dice?

8. Show that the odds are in favour of a person throwing an ace in four throws with a die.

9. If n coins are tossed, what is the chance that there is one and only one head turned up?

10. Find the chances that in three tosses a coin will turn up one tail and two heads, (1) in this order, (2) in any order.

11. A bag contains 3 white and 6 black balls, from which A, B, C draw in order. Find the chances that they draw (1) each black, (2) A and B black and C white, (3) one of them white and the other two black.

12. A collection of 7 letters is made from an alphabet containing 20 consonants and 5 vowels. Find the chance that it contains exactly 3 vowels.

13. There are three tickets in a bag, numbered 1, 2, 3, and a ticket is drawn and put back. If this be done four times, show that it is 41 to 40 that the sum of the numbers drawn is even.

14. A coin is tossed n times, what is the chance of the head turning up an even number of times?

15. If two coins are tossed three times, what are the chances that there will be two heads and four tails? What are the odds against five heads exactly turning up?

16. Find how many odd numbers, taken at random, must be multiplied together, that there may be at least an even chance of the last figure being a 5. (Given $\log_{10} 2 = \cdot 30103$.)

17. A and B play at a game together, A wins 4, and B 3, out of every 7 games. What is A's chance of winning at least 6 in any 7 games?

18. A and B play together, A's skill is to B's as 3 to 5. Find A's chance of winning 3 games out of 5.

Independent Events.

269. *Def.* Events are said to be *independent*, if the occurrence of one does not affect the occurrence of any of the others.

Thus in *Ex.* 1 of Art. 254, a man's being of a certain height, and his having a certain colour of hair, are independent events.

270. Prop. *If* p, q *be the respective chances of two independent events* (A *and* B) *happening on any one of a succession of occasions, when one or both must happen, then the chance of both happening on any given occasion is* pq.

Let $p=\frac{a}{b}$, $q=\frac{c}{d}$.

Consider bd occasions, of these A will occur on ad, on ac of which B will occur; *i.e.* out of bd occasions, A and B concur on ac; $\therefore$ the chance of both happening on the same occasion is $\frac{ac}{bd}=pq$. Q.E.D.

Cor. 1. If $p_1, p_2, \ldots p_r$ are the chances of r independent events happening on any one occasion, then $p_1.p_2 \ldots p_r$ is the chance of their concurrence.

For $p_1.p_2$ is the chance of the concurrence of the first and second; but this concurrence is independent of the happening of the third; $\therefore$ the chance of its happening on the same occasion as the third is $p_1.p_2.p_3$; and so on for any number of events.

Cor. 2. If the chances of the happening of r independent events are each p, then p^r is the chance of their concurrence.

271. Prop. *Also the chance of one, but not both, happening is* p+q−2pq.

For, of the bd occasions considered, A occurs on ad, on ac of which B will occur; $\therefore$ A occurs alone on $ad-ac$; similarly, B occurs alone on $bc-ac$; $\therefore$ either A or B happens alone on $ad+bc-2ac$ occasions; $\therefore$ the chance of one happening alone is

$$\frac{ad+bc-2ac}{bd}=p+q-2pq.$$

Q.E.D.

272. *Ex.* The chance of one man (A) living a year is $\frac{67}{68}$, and of another (B) is $\frac{11}{120}$. What is the chance that they both are alive in a year's time?

Consider 68 people under the same circumstances as A,
and 120 „ „ B,

Then we can form 68×120 pairs such as A and B together, and these are what in Art. 270 we have called occasions.

Now 67×11 of these are pairs of which both members live for another year.

Hence the chance that, of any one pair, both live for a year, is $\frac{67 \times 11}{68 \times 120} = \frac{737}{8160}$.

273. We might have discussed the problem in Art. 271 as follows.

(1) The chance of A happening is p; $\therefore$ (2), by Art. 262, the chance of it not happening is $1-p$.

(3) The chance of B happening is q; $\therefore$ (4), by Art. 262, the chance of it not happening is $1-q$.

Now (1) and (2) are independent of (3) and (4); $\therefore$, Art. 270, the chance of A happening and B not is $p(1-q)$, ... I.
and „ B „ A „ $q(1-p)$, ... II.
But I. and II. are exclusive; $\therefore$, Art. 263, the chance of one or other occurring is $p(1-q)+q(1-p)$.

274. Many of the problems we have already discussed may be treated as belonging to this class.

Thus, from a bag containing 3 red, 4 black, and 2 white balls, what is the chance of drawing first a red and then a white ball, each ball *being put back* after it is drawn?

Since each ball is put back after being drawn, the first drawing does not affect the second.

Hence the drawings are independent events.

Now the chance of a red ball at the first drawing is $\frac{3}{9}$ or $\frac{1}{3}$, and the chance of a white at the second drawing is $\frac{2}{9}$; $\therefore$ the chance required is $\frac{1}{3}\cdot\frac{2}{9}=\frac{2}{27}$.

The chance of a red at the first drawing and a red or black at the second is $\frac{1}{3}\cdot\frac{7}{9}=\frac{7}{27}$.

If we required the chance of a red and white in either order, we have the chance of a red and then a white as $\frac{2}{27}$, similarly

„ „ white „ red $\frac{2}{27}$;

and these two compound events being exclusive (Art. 263), the required chance (*i.e.* of a red and white, or a white and red) is $\frac{2}{27}+\frac{2}{27}=\frac{4}{27}$.

EXAMPLES.—LVII.

Out of every 100,000 girls who attain to 3 years of age, nearly 83,000 attain to 28, and 63,000 to 53.

1. What are the chances that, of two girls, aged respectively 3 and 28, (1) both are alive in 25 years time, (2) the youngest only is alive, (3) the eldest only, (4) both are dead?

2. What are the chances that, of two girls, who attain the age of 3, (1) both live to be 28, (2) both live to be 53?

3. What are the chances that, of two girls, who attain the age of 28, (1) both live to be 53, (2) one only.

4. A card is drawn from a pack and put back, and a card drawn again. What are the chances, (1) that the first was the queen of hearts, (2) that the second was the king of hearts, (3) that the first was the queen *and* the second the king of hearts, (4) that the king and queen of hearts were drawn?

5. What is the chance of throwing sixes with two dice? What is the chance of throwing three?

6. From a bag containing 4 white, 5 red, and 3 black balls, three are drawn, *each as it is drawn being put back.* What are the chances of drawing (1) white, red, black in this order, (2) white, red, black in any order, (3) a white and then two reds?

275. We can now give a shorter proof of the proposition in Art. 265.

Consider each of the sequences of n occasions as a single event.

Now the way, in which any one of these occasions has happened, does not affect the occurrence of any other, *i.e.* they are independent.

Hence the chance that, in any one sequence, the first r occasions should give A, and the remaining $n-r$ should give B, is $(p.p\ldots$ to r factors$)(q.q\ldots$ to $n-r$ factors$)$, Art. 270, Cor. 1,

$$=p^r.q^{n-r}.$$

In the same way we can show that $p^r.q^{n-r}$ is the chance of A and B occurring in any sequence, on r and $n-r$ occasions respectively in any other given order of arrangement. And such arrangements are exclusive and $\dfrac{\underline{|n}}{\underline{|r}\,\underline{|n-r}}$ in number;

$\therefore$, Art. 263, Cor., $\dfrac{\underline{|n}}{\underline{|r}\,\underline{|n-r}}p^rq^{n-r}$ is the required chance.

Cor. A occurs r times at least in every sequence, in which it occurs r, $r+1,\ldots n-1$ or n times. Now all such sequences are exclusive, and $\therefore$ the chance that one or other of such sequences should occur is, Art. 263,

$$p^n+np^{n-1}q+\ldots+\frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}}p^rq^{n-r}.$$

This then is the chance of A occurring r times at least in any sequence of n occasions.

276. By working with the fractional notation we have, up to this point, been able to exhibit in every case the ratio existing between the number of particular events and the number of general events amongst which the former occur.

Also, by starting with a proper number of the general events, we have been able to give exactly the number of them which belonged to the particular class. Thus, in Art. 270, if the number of occasions we first considered had not been a multiple of d, such as bd, we should not have been able to express exactly the number of occasions in which B occurred.

For the future we shall not be so careful to exhibit our work in this form. Thus, when the chance of an event happening is p, we shall consider m occasions on which it may happen; then the number of them, on which it happens, will be expressed by $p.m$.

When we use this notation, we shall assume that m is such that any product, into which it enters, is a whole number.

Compound events, the components being dependent.

277. *Def.* Events are said to be *dependent* when the fact, that one has happened, affects, but does not exclude, the happening of the others. That is, the occurrence of any one event does not prevent the occurrence of any of the other events; but, in some way or other, makes the likelihood of their occurrences different to what it was before.

278. *Ex.* I. There are 5 white and 7 red balls in a bag, what is the chance that a white ball is drawn and then a red, the first ball not being put back?

Let there be m pairs of drawings. Out of the m drawings, which are made first in each pair, $\frac{5}{12}m$ will produce a white ball.

Now, a white having been drawn, there are 11 balls remaining, of which 7 are red; $\therefore$ of the $\frac{5}{12}m$ drawings, which follow, a red will be drawn at $\frac{7}{11}\cdot\frac{5}{12}m$.

Hence, out of m pairs of drawings, there are $\frac{35}{132}m$, at which a white is followed by a red; $\therefore$ the chance required $=\frac{35}{132}$.

Ex. 2. There are three urns, of which, one contains 4 white and 2 red, another 5 white and 1 red, the third 3 white and 4 red, balls; and there is no general reason why one urn should be selected more than another, or one ball from an urn more than any other ball from that urn. Find the chance of a red ball being drawn.

Since all the urns are equally likely to be drawn from, therefore of any number of drawings one-third will be made from each of the urns.

Let there be m drawings; of these $\frac{1}{3}m$ will be made from the first urn.

Now, when a drawing is made from the first urn, all the balls in it are equally likely to be drawn; therefore, of these $\frac{1}{3}m$ drawings from the first urn, $\frac{2}{6}\cdot\frac{1}{3}m$, or $\frac{1}{9}m$, will give a red ball.

Similarly a red will be drawn $\frac{1}{18}m$ times from the second urn, and $\frac{4}{21}m$ from the third.

Hence, out of m drawings, $\frac{1}{9}m+\frac{1}{18}m+\frac{4}{21}m$ will give red balls; $\therefore$ the chance of a red at any one drawing is $\frac{1}{9}+\frac{1}{18}+\frac{4}{21}=\frac{14+7+24}{126}=\frac{45}{126}=\frac{5}{14}$.

279. Prop. *If there be any number of dependent events, such that* p_1 *is the chance of the first happening,* p_2 *the chance that when the first has happened the second will follow,* p_3 *the chance that when the first two have happened the third will follow, and so on, then the chance that they all happen in this order is* $p_1 p_2 p_3 \ldots$

Consider m occasions on which the first may happen, then it happens on $p_1 m$. Of these $p_1 m$ occasions, the second follows on $p_2 p_1 m$, and so on.

Hence out of m occasions on which the first may happen it is followed by the rest in the given order on $p_1 p_2 \ldots m$ occasions; $\therefore$ the chance of this sequence happening is $p_1 p_2 p_3 \ldots$

280. The reader will see that Art. 270 is a particular case of Art. 279. For q being the chance that B happens on *any* occasion, it is, of course, the chance that B shall happen when A happens; $\therefore$ the chance, by Art. 279, that A happens and is accompanied by B, is pq.

281. *Ex.* 1. A and B draw from a bag containing 2 white and 3 black balls, the ball being put back after each drawing, until a white is drawn. What are their respective chances of drawing a white?

The chance that A draws a white at the first time is $\frac{2}{5}$.

The chance that B has a drawing at all is the chance that A draws a black at the first time, *i.e.* $\frac{3}{5}$.

The chance, that, having a drawing, B will draw a white, is $\frac{2}{5}$; $\therefore$, Art. 279, the chance of his drawing a white at the second drawing is $\frac{3}{5} \cdot \frac{2}{5}$.

The chance that A has a second drawing is the chance that A and B both draw black balls, which is $\left(\frac{3}{5}\right)^2$; $\therefore$ the chance that he draws a white at his second drawing is $\left(\frac{3}{5}\right)^2 \cdot \frac{2}{5}$.

Hence A's various chances of drawing a white are

$$\frac{2}{5},\ \left(\frac{3}{5}\right)^2\frac{2}{5},\ \left(\frac{3}{5}\right)^4 \cdot \frac{2}{5},\ \text{etc.},$$

and his drawing a white at any one time excludes the possibility of his doing so at any other; $\therefore$, by Art. 263, his chance of a white ball is $\frac{2}{5}+\left(\frac{3}{5}\right)^2 \cdot \frac{2}{5}+\left(\frac{3}{5}\right)^4\frac{2}{5}+$ etc., *ad infinitum*, $=\frac{5}{8}$.

Similarly B's chance is $\frac{3}{5}\cdot\frac{2}{5}+\left(\frac{3}{5}\right)^3\cdot\frac{2}{5}+$etc.; *ad infin.*$=\frac{3}{8}$.

This result shows that, if a large number of such games be played, then, on the average, 5 out of 8 will be won by the player who begins, and 3 out of 8 by the other.

Here each game constitutes one of the general events, and must belong to one or other of two particular sections, viz., those won by the first player, and those won by the second.

Ex. 2. What would be their respective chances, if a ball when drawn is not put back?

The chance of A drawing a white at first is $\frac{2}{5}$.

If B gets a drawing, there will be 2 white and 2 black balls, hence his chance *then* will be $\frac{1}{2}$;

$\therefore$ B's chance of a white at his first drawing is $\frac{3}{5}\cdot\frac{1}{2}$.

If A gets another drawing, there will be 2 white and 1 black, hence his chance *then* will be $\frac{2}{3}$:

But his chance of getting another drawing is $\frac{3}{5} \cdot \frac{1}{2}$;

$\therefore$ *A*'s chance of a white at his second drawing is $\frac{1}{5}$.

Similarly *B*'s chance of a white at his second drawing is $\frac{3}{5} \cdot \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{10}$.

Therefore *A*'s chance of a white at either drawing is $\frac{2}{5} + \frac{1}{5} = \frac{3}{5}$,

and *B*'s " " $\frac{3}{10} + \frac{1}{10} = \frac{4}{10} = \frac{2}{5}$.

EXAMPLES.—LVIII.

1. There are 3 parcels of books in another room, and a particular book is in one of them. The odds that it is in one particular parcel are 3 to 2; but if not in that one, it is equally likely to be in either of the others. If I send for this parcel, giving a description of it, and the odds that I get the one I describe are 2 : 1, what is the chance of getting the book I want?

2. A certain sum is to be won by the first person who throws head with a penny. If there be m throwers, find the chance of the rth person.

3. *A* and *B* throw for a certain stake, *A* having a die whose faces are marked with the numbers 10, 13, 16, 20, 21, 25, and *B* a die whose faces are marked with the numbers 5, 10, 15, 20, 25, 30. The highest throw to win, and equal throws to go for nothing. Prove that *A*'s chance of winning is $\frac{17}{33}$.

4. A pack of cards is separated into four packets, viz., 13 hearts, 13 spades, 13 clubs, 13 diamonds. What is the chance of drawing the ace of clubs?

5. Four persons throw a die in succession until one throws an ace. What are their respective chances of throwing an ace (1) in the first round, (2) in the second round, (3) at all?

6. Of two urns A and B, A contains 3 white balls, and B 3 black only. A ball is drawn from each and placed in the other. What is now the chance of drawing a black ball from A?

If this interchange of balls be performed three times, what are the chances that B contains (1) 3 white balls, (2) 2 black and 1 white?

7. Out of a pack of 13 cards, numbered 1, . . . 13, what is the chance of drawing first the 1, then the 2, and thirdly the 3?

8. A bag contains 3 white and 6 black balls, what is the chance of drawing one white ball at least in three trials; a ball when drawn not being put back? What is the chance of not drawing a white ball until the fourth trial?

9. Two bags contain each 3 white and 5 black balls, a ball is drawn from one bag, and, if it is white, it is put into the second bag, from which a ball is then drawn. Find the chance of two white balls being drawn.

10. A, B, C throw in this order with 3 dice together, until one of them throws 9 exactly at one throw. Find their respective chances of being the last to throw.

11. One urn contains 5 white and 9 black, another 6 white and 8 black balls. A ball is taken from one and placed in the other. If a ball be now drawn, what is the chance of its being white, it being understood that at each drawing either urn is as likely to be drawn from as the other?

12. Two persons, A and B, play for a stake, throwing each alternately two dice, A commencing. A wins if he throws 6, B if he throws 7, the game ceasing so soon as either event happens. Show that A's chance to B's chance $= 30 : 31$.

Inverse Probabilities.

282. *Ex.* There are 3 bags,

one containing	2	white and	3	red balls,
the second ,,	5	,,	2	,,
the third ,,	4	,,	7	,,

and all the bags are equally likely to be drawn from.

A white ball has been drawn; what is the chance that it was drawn from the first bag?

Let there be m drawings, of these $\frac{1}{3}m$ will be made from the first bag, and of these $\frac{2}{5}\,\frac{1}{3}m$ will give a white ball.

Similarly $\frac{5}{7}\,\frac{1}{3}\,m$ will be the number of occasions, on which a white will be drawn from the second bag; and $\frac{4}{11}\,\frac{1}{3}\,m$ will be the number of occasions, on which a white is drawn from third bag.

Hence, out of every $\left(\frac{2}{5}+\frac{5}{7}+\frac{4}{11}\right)\frac{1}{3}\,m$, *i.e.* $\frac{569}{1155}\,m$ occasions, on which a white is drawn, it will be from the first bag on $\frac{2}{15}\,m$ occasions.

Hence the chance that, a white ball being drawn, it comes from the first bag is $\dfrac{\frac{2}{15}\,m}{\frac{569}{1165}\,m}=\dfrac{2}{\frac{569}{77}}=\dfrac{154}{569}$.

Here the general event was a white ball being drawn, and it could belong to one of three sections, viz., being drawn from the first bag, or from the second, or from the third.

283. *Ex.* A bag contains 3 balls, each of which may be either white or black; and a white ball is drawn. What is the chance that the bag contained 2 white and 1 black?

The bag may either contain (1) all white, or (2) 2 white and 1 black, or (3) 1 white and 2 black, and there is no general *à priori* reason for supposing one assortment to exist rather than another; we therefore *assume* them all three to be equally likely; by this we mean that in any large number of such bags, we should find as many which contained one assortment, as those which contained another.

Hence, out of m drawings from any such large number of bags, $\frac{1}{3}m$ would come from bags of the first class, and then each drawing would give a white ball; also $\frac{1}{3}m$ would come from bags of the second class, and then $\frac{2}{3}\cdot\frac{1}{3}m$ of these would give white balls; and similarly $\frac{1}{3}\cdot\frac{1}{3}m$ would be drawings from bags of the third class which would give white balls.

Hence, out of $\frac{1}{3}m\left(1+\frac{2}{3}+\frac{1}{3}\right)$, *i.e.* $\frac{2}{3}m$, drawings which give white balls, $\frac{2}{9}m$ come from bags of the second class.

Hence the chance that a bag, which yields a white ball, is one of the second class is $\dfrac{\frac{2}{9}m}{\frac{2}{3}m}=\frac{1}{3}$.

284. Prop. *If* $P_1, P_2, \ldots P_n$ *be the probabilities of the existence of* n *causes, such that any one may be followed by a particular event, and* $p_1, \ldots p_n$ *the probabilities that, when the several causes respectively exist, they will be followed by the given*

event, then the probability, that on any occasion, when the event has happened, it has arisen from the r*th cause, is*

$$\frac{P_r.p_r}{P_1p_1+P_2p_2+\ .\ .\ .\ +P_np_n}.$$

Consider m occasions, on which any one of the causes may exist, then the 1st cause exists on P_1m of these occasions,

and " 2nd " P_2m "

" rth " P_rm "

and so on.

Then of the P_1m occasions, on which the first cause exists, the event follows on p_1P_1m, and of the P_rm occasions, on which the rth cause exists, the event follows on p_rP_rm.

Hence the event occurs on $P_1p_1m+P_2p_2m+\ .\ .\ .\ +P_np_nm$ occasions altogether, and of these it arises from the rth cause on P_rp_rm.

Hence the chance, that on any occasion, on which it occurs, it has arisen from the rth cause, is

$$\frac{P_rp_rm}{(P_1p_1+\ .\ .\ .\ +P_np_n)m}=\frac{P_rp_r}{P_1p_1+\ .\ .\ .\ +P_np_n}=\frac{P_rp_r}{\Sigma(Pp)}.$$

285. The values of the chances $P_1, P_2, \ldots P_n$ were obtained independently of the occurrence of the given event, and are called *à priori* chances, whilst $\frac{P_rp_r}{\Sigma(Pp)}$ is called the *à posteriori* probability of the rth cause having existed.

286. To apply this formula to Art. 283. We have

$$P_1=P_2=P_3=\frac{1}{3},$$

$$p_1=1,\ p_2=\frac{2}{3},\ p_3=\frac{1}{3}.$$

Hence the chance required $=\dfrac{\frac{1}{3}\cdot\frac{2}{3}}{1\cdot\frac{1}{3}+\frac{2}{3}\cdot\frac{1}{3}+\frac{1}{3}\cdot\frac{1}{3}}=\frac{1}{3}.$

Also, in Art. 282, $P_1 = P_2 = P_3 = \frac{1}{3}$,

$$p_1 = \frac{2}{5},\ p_2 = \frac{5}{7},\ p_3 = \frac{4}{11};$$

$$\therefore \text{ the chance required} = \frac{\frac{1}{3}\cdot\frac{2}{5}}{\frac{1}{3}\cdot\frac{2}{5} + \frac{1}{3}\cdot\frac{5}{7} + \frac{1}{3}\cdot\frac{4}{11}} = \frac{154}{569}.$$

287. Such problems as the above, from Art. 282, are often said to involve *inverse* probabilities; and we may give the following as the general statement of such problems:—"An event has happened such as might have arisen from different causes; what is the probability that any one specified cause did produce the event, to the exclusion of the other causes?"—De Morgan. Those problems which we considered before Art. 282 are said to involve *direct* probabilities.

EXAMPLES.—LIX.

1. There are two urns, and it is known that one contains 8 white balls and 4 black, and that the other contains 12 black balls and 4 white. From one of these, but it is not known from which, a ball is taken, and is found to be white. Find the chance that it was drawn from the urn containing 8 white balls.

2. A white ball has been drawn from one of two urns A and B. On examination after the drawing, A is found to contain m white and n black balls, and B to contain m' white and n' black balls. What is the probability that the ball was drawn from the urn A?

3. A bag contains 4 balls, each of which may be either black or white.

(1) If on drawing a ball it prove to be white, what is the chance that the rest are white?

(2) If on drawing two balls at once they prove to be white, what is the chance of the remaining two being white?

(3) If on drawing a ball, and, when it is put back, drawing another, we draw white both times, what is the chance that all the balls are white?

4. Into a bag containing either 4 black balls, or 4 white balls, we drop a fifth which is white. On drawing one of the five balls it proves to be white. What is the chance that they are all white?

5. Two bags each contain 5 white and 5 black. Four are taken at random from one and placed in the other. From this latter 8 balls are drawn, viz., 5 white and 3 black. What are the chances that the remainder are (1) all black, (2) 3 white and 3 black?

6. In a bag are 7 balls, of which 2 are white and 3 black for certain, the colour of each of the remaining two is unknown, but it must be either black or white. A ball is drawn and turns out to be white, what is the chance that three white balls are left in the bag?

7. There are two bags, one containing 3 white and 4 black, and the other 5 black and 4 white, and the first is twice as likely to be drawn from as the second.

(1) If a white is drawn, what is the chance that it comes from the first?

(2) If out of three balls drawn from the same bag, two are white, what is the chance that they came from the first bag, (α) when each ball is put back after being drawn, (β) when it is not put back?

288. Oftentimes a problem involves both inverse and direct probabilities, as in the following:—

A bag contains five coins which are known to be either sovereigns or shillings. Two coins are drawn and are seen to be a sovereign and a shilling. If these be replaced and two again drawn, find the chance that they will be a sovereign and a shilling.

We can have four different kinds of bags,

(1) containing 4 sovereigns and 1 shilling,
(2) " 3 " 2 "
(3) " 2 " 3 "
(4) " 1 " 4 " .

We assume that all the bags are *à priori* equally likely.

Now the chance that a sovereign and a shilling come at any drawing, from (1) is $\frac{4}{10}$, from (2) is $\frac{6}{10}$, from (3) is $\frac{6}{10}$, from (4) is $\frac{4}{10}$.

Hence the chance that the bag is of the first kind is $\dfrac{\frac{1}{4}\cdot\frac{4}{10}}{\frac{1}{4}\left\{\frac{4}{10}+\frac{6}{10}+\frac{6}{10}+\frac{4}{10}\right\}}=\frac{1}{5}$, of the second kind $\frac{3}{10}$, of the third kind $\frac{3}{10}$, of the fourth $\frac{1}{5}$.

Hence, Art. 279, the chance that the bag is of the first kind, and that a sovereign and a shilling are drawn again $=\frac{1}{5}\cdot\frac{4}{10}=\frac{4}{50}$.

And the chance that the bag is of the second kind, and that a sovereign and a shilling are drawn again $=\frac{9}{50}$.

And the chance that the bag is of the third kind, and that a sovereign and a shilling are drawn again $=\frac{9}{50}$.

And the chance that the bag is of the fourth kind, and that a sovereign and a shilling are drawn again $=\frac{4}{50}$.

Also these events are exclusive. Hence the required chance $=\frac{4+9+9+4}{50}=\frac{13}{25}$.

EXAMPLES.—LX.

1. From an urn, containing 2 white, and 2 black, balls, 2 balls are drawn at random and placed in a bag. From the bag a ball is drawn at random twice successively, and replaced after each drawing; on each occasion it is found to be white. Prove that the chance, that a white ball will come at the third drawing, is $5:6$.

2. A bag contains 5 balls of unknown colour, one was twice drawn and replaced, and in each case it was white. If two balls be now drawn together, find the chance of both being white.

3. A bag contains 4 white, and 4 black, balls; from these four are drawn at random and placed in another bag; three draws are made from the latter, the ball being replaced after each, and each gives white. Prove that the chance of another drawing giving a black ball is $23:200$.

4. In a bag are four balls, each of which must be either black or white; a white ball is now dropped in, and a ball is drawn, which is found to be white, and put back. Find the chance of now drawing a white ball.

5. It is known that of two purses, one contains 4 sovereigns and the other 4 shillings. A coin (of value unknown) is taken from one and put into the other, and from the latter a coin is drawn and is found to be a sovereign. What is the chance of drawing a sovereign again from the same purse?

6. In an urn are three balls, of which the colour of each is black or white; a ball is drawn and replaced twice, and both times it is white; what are the chances of drawing (1) two white, (2) two black, in two more trials?

7. A purse contains n coins, which are either sovereigns or shillings. A coin is drawn and turns out to be a sovereign. What are the odds that it is the only sovereign?

8. A bag contains 5 white, and 7 black, balls, a second bag contains 6 white, and 10 black, balls. Four balls are taken at

random from the first, and 6 from the second, and are placed in a third bag. If a ball be now drawn from this bag, show that the chance that it will be white is 47 : 120.

289. Explanation of the term "Expectation."

If the probability of drawing a white ball from a bag is 2 : 5, and if I am to receive £1 every time, and nothing unless, I do so, what sum ought I justly to pay, so as ultimately not to gain or lose in the long-run by the transaction?

Evidently I must pay £1 every time I draw a white ball. If however, I draw a *very large number* of times, out of every five drawings, two will, on the average, give white balls. Hence I may pay £2 for *every five drawings* that I have. But I may wish to make a regular payment at *every* drawing, and this would be eight shillings.

Now it seems (Venn's *Logic of Chance*, Ch. 3, § 20) that this average payment, having to be made at each drawing, came to be looked upon, instead of as a mere way of paying £2 for every five drawings, as the actual worth of each particular drawing; or that, as I might expect £2 in the *long* run for every five drawings, it would be the same thing to say that I expected eight shillings at each drawing. This average payment then is said to be the expectation at each drawing.

290. The following is a general *definition* of the term *Expectation.*

If p *represent the chance on a given occasion of a certain event happening,* m *the sum of money to be paid if it happen, then the expectation on that occasion is* p.m. It is always understood that nothing is paid on any occasion when the event does not happen.

For example, in Art. 281, *Ex.* 1, if A were always to begin, and the winner of each game were to receive £2 for it, A must pay £10, and B £6, for every eight games they are allowed to play, and, it would be the same thing, if they paid respectively 50 and 30 shillings for *every* game. Thus 50 and 30 shillings

are called the respective expectations of A and B at the beginning of each game.

Ex. X, Y, Z are three exclusive events; a, b, c the amounts I receive, when they happen, respectively; x, y, z their respective chances of occurrence on any occasion when one of them must happen. Then my expectation on each such occasion is $xa+yb+zc$.

For consider any large number (m) of occasions;

Then X happens on xm of them, and on each of these I receive a; therefore I receive xma altogether on the occasions when X happens.

Similarly I receive ymb, and znc, on the occasions when Y, and Z, happen.

Hence altogether on the m occasions I receive $xma+ymb+znc$.

And this is the same as if, on each occasion, I received $\frac{ma+ymb+zmc}{m}$, *i.e.* $xa+yb+zc$, which is therefore called my expectation on each occasion.

EXAMPLES.—LXI.

1. A purse contains three sovereigns and one shilling. What should be paid for permission to draw one coin from it?

2. A bag contains two sovereigns and three shillings. What should be paid for permission to draw two coins from it at once?

3. What is the worth of a ticket in a lottery of 100 tickets, there being 4 prizes of £50, 5 of £40, and 7 of £30?

4. From a bag containing two guineas, three sovereigns, and five shillings, a person is allowed to take out three of them indiscriminately. Determine the value of his expectation.

5. Three persons throw a die in succession, on condition that he who first throws an ace shall receive £1. What should each pay for his chance?

6. From a bag containing 4 sovereigns and 4 shillings, 4 coins are drawn at random and placed in a purse. Two coins are

drawn out of the purse, and found to be both sovereigns. Show that the probable mean value of the coins left in the bag is $29\frac{1}{3}$ shillings.

7. A man is in the habit of playing at a gaming-table where the odds against him are 11 to 10. He plays on an average four nights a week, and stakes a hundred guineas in the course of the evening. How much per annum is his amusement likely to cost him?

8. A bag contains six shillings and two sovereigns. Find the chances of a person's drawing a shilling at the first, second, or third time, and not before. What is the value of his expectation, if he is allowed to draw till he draws a sovereign?

9. Counters (n) marked with consecutive numbers are placed in a bag, from which a number of counters (m) is to be drawn out at random. Show that, if the drawer is to receive in £1 the sum of the numbers marked on the counters, the value of his expectation will be an arithmetic mean between the greatest and least sums which can be indicated by the number of counters to be drawn.

10. A person throws n coins, and is to take all those that turn up head. He throws again those that turn up tail the first time on the same condition, and so on for r times. Find the value of his expectation, and the chance that all will have turned up head in r throws at most.

11. A die with m faces, marked 1, 2, 3, . . . etc., up to m, is loaded, so that the chance of a face turning up is proportional to the number on that face. A person is to receive as many shillings as the number on the face he throws. Show that the value of his expectation for one throw is $\frac{2m+1}{3}$ shillings.

12. A man is to receive a certain number of shillings; he knows that the digits of the number are 1, 2, 3, 4, 5; but he is ignorant of the order in which they stand; determine the value of his expectation.

13. A and B draw from a bag, in which are three white and

three black balls, for a sum of £1 to be received by the one who first draws a black ball, and A draws first. What are their expectations, (1) when the balls are replaced as they are drawn, (2) when they are not replaced?

14. A purse contains three notes, each of which is either a £5, a £10, or a £20 note. A note is drawn and put back three times, and each time it is found to be a £5. What would you give for the contents of the bag?

291. We will now give a short notice of the manner, in which an attempt has been made, to establish a definite probability of a person's statements being true or false.

It is sometimes said that the chance of a person speaking the truth is $\frac{a}{b}$.

This means that we conclude, from a general estimate of his character, that, out of every b of his statements, a are true, or, that, by making observations on a very large number of statements made by him, or by some other means, it has been ascertained that, on the average, out of every b statements observed, a have been true.

Now it is very unlikely that this could be done, and even if it were, the particular circumstances of the case, to which we might apply it, might differ so widely from those under which he made the observed statements, that this experience would be no safe guide.

Moreover, probabilities do not give any clue to one particular instance, but only as to certain proportions existing amongst a large number of instances; so that it would be of little use applying the theory of probabilities to any one particular statement, unless we were able to apply it to a great number.

As, however, the student will meet with questions in probabilities stated under this form, we subjoin a few examples.

292. *Independent Testimony.*

Ex. 1. A speaks the truth 3 times out of 4, B 4 times out of 5. Find the chance that they will *agree* in their statements, with regard to some event which has happened.

The chance of A asserting that it happened is $\frac{3}{4}$,

" B " " $\frac{4}{5}$;

∴ " both " " $\frac{3}{5}$, Art. 270.

The chance of A asserting that it did not happen is $\frac{1}{4}$,

" B " " $\frac{1}{5}$;

∴ " both " " $\frac{1}{20}$, Art. 270.

But these two concurrences are exclusive; ∴ the chance of one or other is
$$\frac{3}{5}+\frac{1}{20}=\frac{13}{20}.$$

In other words, out of every 20 occasions on which they both make statements, they concur 13 times.

Ex. 2. If they both assert that the event has happened, required the chance that it did happen.

Here, the general events are the various occasions on which they concur; and the particular sections, to which these general events may belong, are, (1) the occasions on which they concur in telling the truth, and (2) those on which they concur in a falsehood. Now out of 13 times on which they concur, they tell the truth 12 times; therefore the chance required is 12 : 13.

293. *Traditionary Testimony.*

If a man makes a statement, purporting to be founded on what another has told him, he is said to have reported truly if he states the matter as it was told him, and to have reported untruly if he states the matter differently.

Ex. In Art. 292, if B report on the statement of A that an event has happened, required the chance that it has happened.

Consider m times that the report is made, on $\frac{3m}{4}$ A has spoken the truth, and on $\frac{4}{5}\cdot\frac{3m}{4}$, or $\frac{3m}{5}$, of these B reports truly.

Again, on $\frac{1}{4}m$ A has lied, and on $\frac{1}{5}\cdot\frac{1}{4}m$ of these, *i.e.* on $\frac{1}{20}m$, B has reported untruly, but on these last occasions, as well as on the former, B has spoken according to the fact.

Hence out of m times on which B makes a report, he speaks according to fact on $\frac{3m}{5}+\frac{1}{20}m$, *i.e.* on $\frac{13m}{20}$.

Therefore the chance, that what he says is according to the fact, is

$$\frac{\frac{13}{20}m}{m}=\frac{13}{20}.$$

Here the general event is B making a report on hearsay from A, which may belong to one of two particular sections, (1) when it is according to the fact, and (2) when it is not according to fact.

294. *Ex.* A bag contains 3 balls, 2 white and 1 black, and a ball is drawn from it. The two persons of Art. 292 concur in saying it is black. What is the probability of its being black?

The *à priori* probability of the ball drawn being white is $\frac{2}{3}$,

and „ „ black „ $\frac{1}{3}$.

Also the probability that, if the ball is white, they concur in denying that it is so, is $\frac{1}{13}$.

And the probability that, if the ball is black, they concur in saying that it is so, is $\frac{12}{13}$.

Hence, Art. 284, the required probability

$$=\frac{\frac{1}{3}\cdot\frac{12}{13}}{\frac{2}{3}\cdot\frac{1}{13}+\frac{1}{3}\cdot\frac{12}{13}}=\frac{12}{14}=\frac{6}{7}.$$

In all examples it is always supposed, that the *à priori* probabilities of an event happening and not happening are equal, unless the contrary is indicated.

EXAMPLES.—LXII.

1. Four judges agree on a verdict; it is known that two are wrong 1 out of 6 times, a third 2 out of 11, and a fourth 1 out of 12. What is the chance that the verdict is right?

2. *A* speaks truly 3 out of 7 times, *B* 5 out of 6, *C* 7 out of 8. What is the chance of a statement being true, which is asserted by *A* and *B* and denied by *C*?

3. *A* speaks the truth 7 times out of 8, *B* 5 out of 6. What is the probability for an event which (1) they both assert, (2) *A* asserts and *B* denies, (3) they both deny, (4) *A* asserts that he heard *B* deny; it being known in (4) that *B* has either denied or asserted the fact, and *A* can only derive information relating to the event from *B*?

4. *A*'s truthfulness is represented by $\frac{3}{5}$, *B*'s by $\frac{6}{7}$, *C*'s by $\frac{4}{9}$. What is the probability of the truth of a statement which *A* and *B* assert that *C* denied?

5. It is 3 to 1 that *A* speaks the truth, 5 to 1 that *B* does. They agree in asserting that a white ball has been drawn from a bag containing 2 white and 3 red and no others. What is the probability that this statement is true?

XXII

Tables of Mortality, etc.

295. Observations have been made on the number of persons living, born, and dying in each of a succession of years, at certain places, and records have been kept of the results. These records are called *Tables of Mortality.*

296. We give below extracts from the *Carlisle* Table of Mortality, which was formed by Mr. Milne from observations made at Carlisle by Dr. Heysham in the years 1779-1787.

Age.	Living.	Decr.	Age.	Living.	Decr.	Age.	Living.	Decr.
0	10000	1539	30	5642	57	60	3643	122
1	8461	682	31	5585	57	61	3521	126
2	7779	505	32	5528	56	62	3395	127
3	7274	276	33	5472	55	63	3268	125
4	6998	201	34	5417	55	64	3143	125
5	6797	121	35	5362	55	65	3018	124

The numbers in the *second* column (headed Living) indicate how many persons, on the average, out of every 10,000 born, were alive at the ages given, in the same horizontal rows, in the first column (headed Age).

Thus out of every 10,000 born, 5417 reached the age of 34, and of these 3395 reached the age of 62.

The numbers in the *third* column (headed Decr., for Decrements), indicate how many, out of every 10,000 born, died in the

course of the next years, after reaching the ages given in the same horizontal rows in the first column.

Thus 1539 died in their first year, and 56 in their 33rd.

297. In the same way the *Northampton* Table was formed from observations made at that place in the years 1741-1780.

The various Life Insurance Offices have also formed similar tables, founded on their experiences amongst their own clients.

298. We will now give some examples of a few of the principal ways, in which these tables can be employed.

We shall assume that the tables hold good, for people living in most parts of England, and at times other than those at which the observations were made.

299. *Ex.* 1. Of the people who reach 33 years of age, what proportion reaches 60?

This proportion is denoted by ${}_{27}p_{33}$.

Out of 5472 living at 33, 3643 survive till they are 60.

Hence the required proportion is $\frac{3643}{5472}$.

Ex. 2. What is the chance that two people, now alive, both die within 30 years, one being 30 and the other 32 years of age?

This chance is denoted by $|{}_{30}q_{3032}$; *i.e.* $|{}_{n}q_{xy}$ denotes the chance of two people, aged x and y years, respectively, both dying within n years.

Out of 5642 people reaching 30 years of age, 5642−3643, or 1999, die before they are 60, that is, within the next 30 years; and of 5528 reaching 32 years, 2133 die within the next 30 years.

Hence the chances of *each* dying within 30 years are respectively represented by $\frac{1999}{5642}$ and $\frac{2133}{5528}$. Now these are independent events; therefore the chance that *both* die within 30 years is represented by $\frac{1999}{5642} \times \frac{2133}{5528}$.

300. The *mean*, or *average*, *duration* of life beyond a given age, or, as it is sometimes called, the *expectation of life* at that age, is the average of the number of years enjoyed beyond that age, by those who reach it.

Of two men, who are alive at the beginning of the year, suppose one lives 3 and the other 9 months, then the whole time enjoyed by the two together is 1 year, and the mean of the time is $\frac{1}{2}$ year.

301. *Let* l_m *denote the number of persons who complete their* m*th year*, i.e., *the number in the second column in the table, on a level with* m *in the first. To find the average duration of life of these* l_m *persons.*

Now $l_m - l_{m+1}$ denotes the number of those who die in their $\overline{m+1}$th year, *i.e.*, the number in the third column on the same level.

We will suppose these persons to die so that, for every death at any distance from the beginning of the year, there is one at the same distance from the end.

Then the time enjoyed in that year by each *pair*, who die, is 1 year, Art. 300; hence the average time enjoyed by *each* of the $l_m - l_{m+1}$ *persons* is $\frac{1}{2}$ year, and $\therefore$ the whole time enjoyed by them is $\frac{l_m - l_{m+1}}{2}$ years.

And the time enjoyed by the l_{m+1} survivors is of course l_{m+1} years.

Hence the time enjoyed in their $\overline{m+1}$th year by the l_m, who reach m years of age, is $\frac{l_m - l_{m+1}}{2} + l_{m+1}$, or $\frac{l_m + l_{m+1}}{2}$ years.

In the same way we can show that the times enjoyed by them in their $\overline{m+2}$th, $\overline{m+3}$rd, etc. years, are respectively

$$\frac{l_{m+1} + l_{m+2}}{2}, \quad \frac{l_{m+2} + l_{m+3}}{2}, \text{ etc.}$$

Let the $(m+z)$th be the greatest age given in the table, so that $l_{m+z+1} = 0$,

Then the total number of years enjoyed by the l_m persons is

$$\frac{l_m+l_{m+1}}{2}+\frac{l_{m+1}+l_{m+2}}{2}+\text{ etc. }+\frac{l_{m+z-1}+l_{m+z}}{2}+\frac{l_{m+z}}{2},$$

$$=\frac{l_m}{2}+l_{m+1}+l_{m+2}+\ldots+l_{m+z-1}+l_{m+z}.$$

Hence the average number of years enjoyed by each is

$$\tfrac{1}{2}+\frac{l_{m+1}+l_{m+2}+\ \ldots\ +l_{m+z-1}+l_{m+z}}{l_m}.$$

This average duration of life at the age of m years is denoted by $\mathring{e}_m$.

302. *To find the payment* ($\mathrm{a_m}$) *that must be made to an office, by a person* m *years of age, to obtain an annuity of* £1 *per annum, for the rest of his life, to commence a year after the payment is made.*

Let each of the l_m persons, whom the table represents as being alive at the age of m years, enter into the same agreement with the office.

Suppose interest to be paid yearly (see Chap. XIII.)

At the end of 1 year the office must pay £l_{m+1}, of which the present value is $l_{m+1}(1+r)^{-1}$;

At the end of 2 years the office must pay £l_{m+2}, of which the present value is $l_{m+2}(1+r)^{-2}$; etc., etc.

At the end of z years the office must pay £l_{m+z}, of which the present value is $l_{m+z}(1+r)^{-z}$.

Hence the present value of the debt incurred by the office is

$$l_{m+1}(1+r)^{-1}+l_{m+2}(1+r)^{-2}+\ \ldots\ +l_{m+z}(1+r)^{-z};$$

and, this having to be paid for equally by the l_m persons, each must pay

$$\frac{l_{m+1}(1+r)^{-1}+l_{m+2}(1+r)^{-2}+\ \ldots\ +l_{m+z}(1+r)^{-z}}{l_m}$$

$$=\frac{l_{m+1}(1+r)^{-\overline{m+1}}+l_{m+2}(1+r)^{-\overline{m+2}}+\ \ldots\ +l_{m+z}(1+r)^{-\overline{m+z}}}{l_m(1+r)^{-m}}.$$

This is the sum to be paid for an annuity of £1, in order that the office in the long-run may neither gain nor lose.

The numerator of this fraction is denoted by N_m, and is to be found by multiplying each number given in the column "*Living*" in a Table of Mortality, with the number in the corresponding place of a table of present values of £1 (see Art. 132), and adding these products together, from that corresponding to the greatest age given in the Table of Mortality down to the product for the age $\overline{m+1}$ inclusive.

A table has been formed of the values of N_m for all values of m from 0 to 103, and for various rates of interest from 3 to 10 per cent. Jones on *Annuities*, vol. i., Tables XI.-XVIII.

303. When an engagement is entered into to secure the payment of a stipulated amount upon the death of an individual, in consideration for a single, or annual, payment, such a transaction is called an assurance on the life of the individual.

The stipulated amount is called the *sum assured*, and the single, or annual, payment is called the *premium*.

304. *To find the annual premium* ($\hat{\omega}_m$ or P_m) *to be paid, for the assurance of £1, on the life of a person, aged* m *years when the first premium is paid.*

Let the lives of each of l_m persons aged m years be assured at the same office for £1, and suppose that each assurance is paid off at the end of the year in which the person, whose life is assured, dies.

Let d_{m+1} denote the number of persons who attain m but not $m+1$ years of age, then $d_{m+1}=l_m-l_{m+1}$, and is evidently the number placed on a level with m in the column "*Decr.*" of a table of mortality. Thus $d_{32} = 56$.

Then the office would have to pay £d_{m+1}, £d_{m+2}, . . . £d_{m+z+1} at the end of 1, 2, 3 . . . $\overline{z+1}$ years respectively; hence, as in Art. 302, the present value of the debt it incurs is represented by $d_{m+1}(1+r)^{-1}+ \ . \ . \ . \ +d_{m+z+1}(1+r)^{-\overline{z+1}}$.

And this has to be met by an annual payment of £P_m by each

of the l_m persons, as long as he, or she, shall be alive, at the time for the annual payment.

Hence the office receives

$P_m l_m$ at once, and $P_m l_{m+1}$, $P_m l_{m+2}$, ... $P_m l_{m+z}$ at the end of 1, 2, 3, ... z years respectively.

The present value of all these sums is

$A_m\{l_m+l_{m+1}(1+r)^{-1}+l_{m+2}(1+r)^{-2}+ \ . \ . \ . \ +l_{m+z}(1+r)^{-z}\}$;

and this, in order that the office may neither gain nor lose in the long-run, must be equal to the present value of the debt incurred; we have, therefore,

$$P_m=\frac{d_{m+1}(1+r)^{-1}+ \ . \ . \ . \ +d_{m+z+1}(1+r)^{-\overline{z+1}}}{l_m+l_{m+1}(1+r)^{-1}+ \ . \ . \ . \ +l_{m+z}(1+r)^{-z}}$$
$$=\frac{d_{m+1}(1+r)^{-\overline{m+1}}+ \ . \ . \ . \ +d_{m+z+1}(1+r)^{-\overline{m+z+1}}}{l_m(1+r)^{-m}+l_{m+1}(1+r)^{-\overline{m+1}}+ \ . \ . \ . \ +l_{m+z}(1+r)^{-\overline{m+z}}},$$

by multiplying each term of the numerator and denominator by $(1+r)^{-m}$.

Call this last numerator M_m, and we have

$$P_m=\frac{M_m}{N_{m-1}}.$$

The values of M_m for different values of m, are calculated in the same way as the values of N_m, except that we use the column *Decr.* instead of the column *Living.*

The values of M_m for all values of m from 0 to 104, and for various rates of interest from 3 to 6, are given in vol. i. of Jones's *Treatise on Annuities*, Tables XI.-XVI.

See also, "The Mortality Experience of Life Assurance Companies," collected by the Institute of Actuaries, London, 1869; and "Tables deduced from the Mortality Experience of Life Assurance Companies," London, 1872.

EXAMPLES.—LXIII.

1. The present value of an annuity of £100 on the life of a person aged 21 is, by the Carlisle Tables of Mortality, reckoning

3 per cent., £2150. If, out of every 10 children born, 6 reach the age of 21, what sum ought to be paid down immediately on the birth of a child in order to secure it an annuity of £100 on its reaching 21, the deposit being forfeited if the child dies previously?

2. A person who has a life annuity (a) wishes to secure to his family after his death an income equal to his present expenditure. What portion of his income must be paid in the way of annual premium to an assurance office?

3. Prove by general reasoning, or otherwise, that the present value of a life annuity to A on the death of B, is equal to the present value of the same quantity to A commencing immediately, minus the present value of the same to continue as long as both live.

4. If A_p denote the value of an annuity to last during the joint lives of p persons of the same age, prove that the value of an equal annuity, to continue so long as there is a survivor out of n persons of that age, may be found by means of tables giving the values of A_p from the formula

$$nA_1 - n\frac{n-1}{2}A_2 + n.\frac{n-1}{2}.\frac{n-2}{3}A_3 - \text{etc.} \pm A_n.$$

XXIII

Determinants.

305. If, having given the system of equations,

$$a_1x+b_1y+c_1z=d_1$$
$$a_2x+b_2y+c_2z=d_2$$
$$a_3x+b_3y+c_3z=d_3,$$

we solve for x, we obtain

$$x\{a_1(b_2c_3-b_3c_2)+a_2(b_3c_1-b_1c_3)+a_3(b_1c_2-b_2c_1)$$
$$=d_1(b_2c_3-b_3c_2)+\text{ etc.} \qquad \text{(App. Art. 1-3.)}$$

The expression, which here appears as the coefficient of x, (call it A) occurs very frequently in mathematical investigations, as well as many others of greater length, but of a similar character. We shall therefore explain a system of notation, by which such expressions can be represented in a more compact form.

306. It will be seen that A consists of the algebraic sum of all the terms, which can be obtained from the square

$$\begin{matrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3, \end{matrix}$$

by observing the following rules:—

1° Pick out three symbols (such as a_1), so that each comes from a different horizontal row and vertical column to either of the other two, and multiply them together.

Thus a_2, b_3, c_1 are three symbols which will be taken together.

2° Prefix the sign $+$ or $-$ to each product, according as it can be obtained from the product $a_1\ b_2\ c_3$ by an even or odd number of interchanges of the suffixes.

Thus to obtain $a_2\ b_3\ c_1$ from $a_1\ b_2\ c_3$, we interchange the 2 and 3 under b and c, producing $a_1\ b_3\ c_2$, and now interchange

the 1 and 2 underneath the a and c, producing $a_2\ b_3\ c_1$; thus we have had *two* interchanges, and ∴ we prefix the sign $+$.

So also the product $a_1\ b_3\ c_2$ would have $-$ prefixed.

307. Hence the square has been subjected to the following definite series of operations:—

I. Multiply together every set of three symbols picked out as in 1°,

II. Prefix the sign $+$ or $-$ according to rule in 2°,

III. Add all such products together.

When such a square has to be subjected to these operations, we indicate the fact by placing a bar on each side of it.

Thus,
$$\begin{vmatrix} a_1, & b_1, & c_1, \\ a_2, & b_2, & c_2, \\ a_3, & b_3, & c_3, \end{vmatrix} \quad . \quad . \quad . \quad (B).$$

Any expression, which is capable of being written down in this manner, is called a *Determinant.*

A and B are said to be different forms of the same determinant.

Each of the symbols a_1, etc., is called a *constituent* of the determinant.

Each of the products, with the proper sign prefixed, is called an *element* of the determinant.

Since, from the diagonal element $+a_1\ b_2\ c_3$ we obtain all the others by proper interchanges of suffixes, it is called the *principal element*, and the determinant is often indicated thus,
$$\Sigma(\pm a_1\ b_2\ c_3), \quad . \quad . \quad . \quad (C),$$
this being a form which takes up much less room in writing and printing.

Such a determinant as we have been considering is said to be of the *third* order, there being *three* factors in each element.

The forms of a determinant of the second order, corresponding to A, B, C, are
$$a_1b_2-a_2b_1, \qquad \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}, \qquad \Sigma(\pm a_1b_2).$$

308. If we interchange two columns of a determinant (of the third order), we obtain another differing from it in sign only. For

$$\begin{vmatrix} b_1 & a_1 & c_1 \\ b_2 & a_2 & c_2 \\ b_3 & a_3 & c_3 \end{vmatrix} = b_1(a_2c_3-a_3c_2)+b_2(a_3c_1-a_1c_3)+b_3(a_1c_2-a_2c_1)$$

$$=-\{a_1(b_2c_3-b_3c_1)+a_2(b_3c_1-b_1c_3)+a_3(b_1c_2-b_2c_1)\}$$

$$=-\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

Similarly $\Sigma(\pm a_1c_2b_3)=-\Sigma(\pm a_1b_2c_3)$.

Cor. $\Sigma(\pm b_1c_2a_3)=-\Sigma(\pm b_1a_2c_3)=\Sigma(\pm a_1b_2c_3)$,

and $\Sigma(\pm c_1a_2b_3)=-\Sigma(\pm a_1c_2b_3)=\Sigma(\pm a_1b_2c_3)$.

309. It will be observed that we may express the solution of the equations in Art. 305 thus,

$$x=\frac{\Sigma(\pm d_1b_2c_3)}{\Sigma(\pm a_1b_2c_3)},\quad y=\frac{\Sigma(\pm d_1c_2a_3)}{\Sigma(\pm b_1c_2a_3)} \quad z=\frac{\Sigma(\pm d_1a_2b_3)}{\Sigma(\pm c_1a_2b_3)}$$

$$\qquad\qquad =\frac{\Sigma(\pm a_1d_2c_3)}{\Sigma(\pm a_1b_2c_3)},\quad =\frac{\Sigma(\pm a_1b_2d_3)}{\Sigma(\pm a_1b_2c_3)}.$$

310. Again

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1(b_2c_3-b_3c_2)+a_2(b_3c_1-b_1c_3)+a_3(b_1c_2-b_2c_1)$$

$$=a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}-a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}+a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}.$$

Thus a determinant of the third order is the sum of the products of each constituent of its first column into the *minor* determinant, formed by omitting the row and column in which that constituent occurs, the sign $+$ or $-$ being placed before each product according as the constituent in it from the first column comes from an odd or even horizontal row.

311. This will be useful in finding the value of any given determinant. Thus

$$\begin{vmatrix} 1 & 2 & 5 \\ 2 & 4 & 2 \\ 3 & 3 & 3 \end{vmatrix} = 1\begin{vmatrix} 4 & 2 \\ 3 & 3 \end{vmatrix} - 2\begin{vmatrix} 2 & 5 \\ 3 & 3 \end{vmatrix} + 3\begin{vmatrix} 2 & 5 \\ 4 & 2 \end{vmatrix}$$

$$= 1(12-6) - 2(6-15) + 3(4-20)$$

$$= 6 + 18 - 48$$

$$= -24.$$

312. We have

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & a_1 & b_1 \\ 0 & a_2 & b_2 \end{vmatrix} = 1\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} - 0\begin{vmatrix} 0 & 0 \\ a_2 & b_2 \end{vmatrix} + 0\begin{vmatrix} 0 & 0 \\ a_1 & b_1 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$$

Thus a determinant of the second order can be written down in the form of a determinant of the third order.

313. Again

$$\begin{vmatrix} ma_1 & b_1 & c_1 \\ ma_2 & b_2 & c_2 \\ ma_3 & b_3 & c_3 \end{vmatrix} = ma_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - ma_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + ma_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

$$= m\left\{ a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \right\}$$

$$= m\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

Thus

$$\begin{vmatrix} 4 & 1 & 5 \\ 6 & 2 & 6 \\ 8 & 3 & 4 \end{vmatrix} = 2\begin{vmatrix} 2 & 1 & 5 \\ 3 & 2 & 6 \\ 4 & 3 & 4 \end{vmatrix} = 2\left\{ 2\begin{vmatrix} 2 & 6 \\ 3 & 4 \end{vmatrix} - 3\begin{vmatrix} 1 & 5 \\ 3 & 4 \end{vmatrix} + 4\begin{vmatrix} 1 & 5 \\ 2 & 6 \end{vmatrix} \right\}$$

$$= 2\{2(8-18) - 3(4-15) + 4(6-10)\}$$

$$= 2\{-20 + 33 - 16\}$$

$$= -6.$$

We have dwelt thus at length on determinants of the third order, for the purpose of familiarizing the student with the notation and nomenclature of the subject. For the same purpose we subjoin the following examples.

EXAMPLES.—LXIV.

1. Find the numerical values of the following :—

(1) $\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix}$ (2) $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$ (3) $\begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix}$

(4) $\begin{vmatrix} 1 & 4 & 5 \\ 0 & 1 & 2 \\ 3 & 5 & 4 \end{vmatrix}$ (5) $\begin{vmatrix} 2 & 2 & 2 \\ 3 & 1 & 1 \\ 4 & 2 & 1 \end{vmatrix}$ (6) $\begin{vmatrix} 0 & 2 & 3 \\ 5 & 0 & 4 \\ 3 & 1 & 0 \end{vmatrix}$

2. Show that $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$;

or, in other words, that the value of a determinant is not altered by changing columns into rows, and *vice versâ*.

It will be observed that this justifies the notation (C) in Art. 307, for by that both the above determinants are represented by $\Sigma(\pm a_1 b_2 c_3)$.

3. Prove that

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & b_1 & c_1 \\ \alpha_2 & b_2 & c_2 \\ \alpha_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1+\alpha_1, & b_1 & c_1 \\ a_2+\alpha_2, & b_2 & c_2 \\ a_3+\alpha_3, & b_3 & c_3 \end{vmatrix}.$$

In other words, if each term of the first column be resolved into the sum of two others, the determinant can be resolved into the sum of two others.

4. Prove that, if each term of a column or row be resolved into two others, the determinant can be resolved into the sum of two others.

5. Prove that, if any two columns, or two rows, be the same, the determinant vanishes.

6. Show that

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix} = \begin{vmatrix} a_1\alpha_1+b_1\beta_1, & a_2\alpha_1+b_2\beta_1 \\ a_1\alpha_2+b_1\beta_2, & a_2\alpha_2+b_2\beta_2 \end{vmatrix}.$$

7. Prove that

$$\begin{vmatrix} 1, & a_1+a_2, & a_1a_2 \\ 1, & b_1+b_2, & b_1b_2 \\ 1, & c_1+c_1, & c_1c_2 \end{vmatrix} = (a_1-b_2)(b_1-c_2)(c_1-a_2) + (a_2-b_1)(b_2-c_1)(c_2-a_1).$$

8. Find the numerical values of

$$\begin{vmatrix} 4 & 1 & 5 \\ 2 & 3 & 0 \\ 6 & 2 & 1 \end{vmatrix}, \text{ and } \begin{vmatrix} 6 & 1 & 0 \\ 0 & 2 & 5 \\ 10 & 3 & 1 \end{vmatrix}.$$

9. Prove that

$$\begin{vmatrix} a_1 & mb_1 & c_1 \\ a_2 & mb_2 & c_2 \\ a_3 & mb_3 & c_3 \end{vmatrix} = m\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ ma_2 & mb_2 & mc_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

10. Find the numerical values of the following:—

(1) $\begin{vmatrix} 1 & 4 & 0 \\ 2 & 6 & 5 \\ 3 & 2 & 1 \end{vmatrix}$ (2) $\begin{vmatrix} 2 & 4 & 6 \\ 5 & 2 & 1 \\ 1 & 3 & 2 \end{vmatrix}$ (3) $\begin{vmatrix} 2 & 4 & 2 \\ 8 & 3 & 0 \\ 6 & 0 & 5 \end{vmatrix}$ (4) $\begin{vmatrix} 1 & 3 & 2 \\ 2 & 6 & 8 \\ 5 & 9 & 4 \end{vmatrix}$.

11. Prove $\begin{vmatrix} A & B & D \\ B & C & E \\ D & E & F \end{vmatrix} = ACF+2EDB-AE^2-CD^2-FB^2.$

The above is called a *symmetrical* determinant.

12. Prove that $\begin{vmatrix} 0 & z & y \\ z & 0 & x \\ y & x & 0 \end{vmatrix} = 2xyz.$

13. Prove that $\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = 2abc(a+b+c)^3.$

314. We shall now give more general definitions and theorems.

Let there be n^2 symbols, arranged in a square of n vertical columns and n horizontal rows.

Each of these n^2 symbols is called a *constituent*.

Thus,

$$\begin{array}{cccccc} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & & a_{n,n} \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ a_{n,1} & a_{n,2} & \cdots & \cdots & \cdots & a_{n,n} ; \end{array}$$

so that $a_{p,q}$ denotes the constituent belonging to the pth row and qth column.

Then we give the name of *determinant* to the algebraic sum of all possible products of n constituents, one of which is taken from each row and each column, the sign $+$ or $-$ being prefixed to each product, according as it can be derived from the *diagonal* product $a_{1,1}a_{2,2}a_{3,3}\ \ldots\ a_{n,n}$ by an even or odd number of interchanges amongst the suffixes indicating the *columns*, *i.e.* in this case the second suffixes.

315. We denote the determinant thus,

$$\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,n} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,n} \end{vmatrix}, \text{ or thus, } \Sigma \pm a_{11}\, a_{22} \ldots a_{nn}.$$

316. Each term of a determinant is called an *element*, and if n be the number of rows, or of columns, and $\therefore$ also of constituents multiplied together to make an element, the determinant is said to be of the nth *order*.

317. The *diagonal* element $a_{11}\, a_{22} \, . \, . \, . \, a_{nn}$ is called the *principal* element, or term.

[*N.B.*—A determinant of the first order consists of only one element, thus $| \, a_{11} \, |$, and $\therefore$ is $= a_{11}$.]

318. Instead of the notation for constituents given above, a shorter form is more frequently adopted; thus the constituent of the pth row, and qth column is denoted by (p, q), and the determinant then appears in either of the forms,

$$\begin{vmatrix} (1,1)(1,2)(1,3) \, . \, . \, . \, . \, . \, . \, (1,n) \\ (2,1)(2,2)(2,3) \, . \, . \, . \, . \, . \, . \, (2,n) \\ . \quad . \quad . \quad . \quad . \quad . \\ . \quad . \quad . \quad . \quad . \quad . \\ . \quad . \quad . \quad . \quad . \quad . \\ (n,1)(n,2)(n,3)(n,4) \, . \, . \, . \, (n,n) \end{vmatrix}$$

or $\Sigma \pm (1, 1)(2, 2)(3, 3) \, . \, . \, . \, (n, n)$.

In the symbols a_{pq} and (p, q), for shortness, we speak of p as the horizontal index and of q as the vertical index.

319. The constituents (p, q) and (q, p) are said to be conjugate to one another, and, if $(p, q) = (q, p)$ for all values of p and q, the determinant is said to be *symmetrical;* see Ex. LXIV. 11.

320. Any determinant formed by omitting *any* r rows and *any* r columns is called an rth *minor* of the original determinant.

Examples. $$\begin{vmatrix} (2,2)(2,3) \, . \, . \, . \, (2,n) \\ (3,2)(3,3) \, . \, . \, . \, (3,n) \\ . \quad . \quad . \quad . \\ . \quad . \quad . \quad . \\ . \quad . \quad . \quad . \\ (n,2)(n,3) \, . \, . \, . \, (n,n) \end{vmatrix}$$ omitting the first row and first column, call it $A_{1,1}$,

$$\left|\begin{matrix} (1,1)(1,2)(1,3)(1,5) & \ldots & (1,n) \\ (3,1)(3,2)(3,3)(3,5) & \ldots & (3,n) \\ (4,1)(4,2)(4,3)(4,5) & \ldots & (4,n) \\ (5,1)(5,2)(5,3)(5,5) & \ldots & (5,n) \\ \cdot\quad\cdot\quad\cdot\quad\cdot & \cdots & \cdot \\ \cdot\quad\cdot\quad\cdot\quad\cdot & \cdots & \cdot \\ \cdot\quad\cdot\quad\cdot\quad\cdot & \cdots & \cdot \\ (n,1) & \ldots\ldots\ldots\ldots & (n,n) \end{matrix}\right|$$ omitting the 2nd row and fourth column, call it $A_{2,4}$,

are *first* minors; and

$$\left|\begin{matrix} (1,1) & (1,2) & (1,4) & (1,6) & . & (1,n) \\ (3,1) & (3,2) & (3,4) & (3,6) & . & (3,n) \\ (4,1) & (4,2) & (4,4) & (4,6) & . & (4,n) \\ (5,1) & (5,2) & (5,4) & (5,6) & . & (5,n) \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ (n-1,1) & (n-1,2) & (n-1,4) & (n-1,6) & . & (n-1,n) \end{matrix}\right|$$ omitting the 2nd and nth rows, and the 3rd and 5th columns,

is a *second* minor, of the determinant in Art. 318.

In general, A_{pq} will denote the *first* minor formed by omitting the pth row and qth column. Its principal element is

$$(1, 1)\ldots(p-1, p-1)(p+1, p)(p+2, p+1)\ldots$$
$$\ldots(q, q-1)(q+1, q+1)\ldots(n, n),$$

or $(1, 1)\ldots(q-1, q-1)(q, q+1)(q+1, q+2)\ldots$
$$\ldots(p-1, p)(p+1, p+1)\ldots(n, n),$$

according as p is $<$, or $>$, q.

For when $p<q$, the pth row being absent, the $(p+1)$th takes its place, and the $(p+2)$th takes the former place of the $(p+1)$th, and so on; hence the suffix for the row is one in advance of the suffix for the column, which cuts it on the diagonal, until the qth column being absent, the $(q+1)$th takes its place, and cuts the diagonal on the same constituent as the original $(q+1)$th row.

Similarly the student will understand the correctness of the form, which we have written down, for the principal element of A_{pq}, when $p>q$.

321. PROP. I. *If from a given determinant we form another, of which each row is the same as the corresponding column of the first, and* vice versâ, *the two determinants are equal.*

Or

$$\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \ldots & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & \ldots & a_{2,n} \\ a_{3,1} & \cdot & \cdot & \cdot\,\cdot\,\cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot\,\cdot\,\cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot\,\cdot\,\cdot & \cdot \\ a_{n,1} & a_{n,2} & a_{n,3} & \ldots & a_{n,n} \end{vmatrix} = \begin{vmatrix} a_{1,1} & a_{2,1} & a_{3,1} & \ldots & a_{n,1} \\ a_{1,2} & a_{2,2} & a_{3,2} & \ldots & a_{n,2} \\ a_{1,3} & \cdot & \cdot & \cdot\,\cdot\,\cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot\,\cdot\,\cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot\,\cdot\,\cdot & \cdot \\ a_{1,n} & a_{2,n} & a_{3,n} & \ldots & a_{n,n} \end{vmatrix}.$$

For their principal elements are the same.

Now in the first determinant suppose that we obtain from the principal element, by a certain series of interchanges of second suffixes, the element $a_{1,\alpha}a_{2,\beta}\ldots a_{n,\nu}$.

In the second determinant it is the first suffixes which mark the columns, $\therefore$ *they* must be interchanged. Write the principal element of the second determinant thus, $a_{\alpha,\alpha}a_{\beta,\beta}\ldots a_{\nu,\nu}$, *i.e.* alter the order in which the constituent factors of the principal element are written down. Interchange the first suffixes till we arrive at $a_{1,\alpha}a_{2,\beta}\ldots a_{n,\nu}$.

It will be seen that we have gone through the same series of interchanges, as in the case of the first determinant, only in the reverse order. For in the first case we interchanged the suffixes $1, 2, 3, \ldots n$ till we arrived at $\alpha, \beta, \ldots \nu$; in the second case we interchanged the suffixes $\alpha, \beta, \ldots \nu$ till we arrived at $1, 2, \ldots n$.

Hence the number of interchanges is the same in both cases.

Therefore in one determinant the elements are the same, in form and sign, as in the other determinant.

Hence the two determinants are identically equal.

This proposition is generally stated thus: "*The value of a determinant is not altered, if the rows are changed into corresponding columns, and* vice versâ."

322. Prop. 2. *If from a given determinant* (D), *we form another* (D′), *by interchanging two columns* (*or two rows*), *then* D′=−D.

Let $D \equiv \Sigma \pm (1, 1)(2, 2)(3, 3) \ldots (n, n)$.

Let the pth and qth columns be interchanged, and let $q > p$, then

$$D' \equiv \Sigma \pm (1, 1)(2, 2) \ldots (p, q) \ldots (q, p) \ldots (n, n);$$

∴ the principal element of D' can be obtained from the principal element of D by the single interchange of the second suffixes p and q.

Hence each element in D' occurs in D, but can be obtained from its principal by *one* more interchange than when it occurs in D, namely, by passing from the principal element of D' through that of D.

Hence D and D' are the same, element for element, only with opposite signs.

By Prop. 1. the same is true when rows are interchanged instead of columns.

This proposition is often stated thus: *If two rows, or two columns, of a determinant are interchanged, the sign of the determinant is changed.*

Cor. 1. If from a determinant D we form another D', by x interchanges amongst the rows or columns, then $D' = (-1)^x D$.

For at each interchange we obtain a new determinant exactly the same as D, except that the sign of the whole is altered at each interchange.

Cor. 2. If two rows, or two columns, are the same, the determinant vanishes.

For if D be the determinant, on interchanging the two rows, or columns, we obtain $-D$ by the Prop.

But this interchange of two things identically the same cannot affect the value of the determinant;

$$\therefore\ D = -D, \quad \text{or } D = 0.$$

COR. 3.

$$\begin{vmatrix} (1,1)(1,2)\dots(1,p-1)(1,p)\dots(1,n) \\ (2,1)\dots\dots\dots(2,p-1)(2,p)\dots(2,n) \\ \cdot\quad\cdot\;\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ \cdot\quad\cdot\;\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ \cdot\quad\cdot\;\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ (n,1)\dots\dots\dots(n,p-1)(n,p)\dots(n,p) \end{vmatrix} = (-1)^{p-1} \begin{vmatrix} (1,p)(1,1)\dots(1,p-1)\dots(1,n) \\ (2,p)(2,2)\dots(2,p-1)\dots(2,n) \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ (n,p)(n,1)\dots(n,p-1)\dots(n,n) \end{vmatrix}.$$

For to obtain the right-hand determinant from the left-hand one, we must interchange, in succession, the original pth column with the $(p-1)$th, . . . , the second, and the first, making in all $p-1$ interchanges;

$\therefore$ the final determinant $=(-1)^{p-1}\times$ original determinant,

or „ original „ $=(-1)^{p-1}\times$ final „ .

323. PROP. 3. *To multiply every constituent of any one row (or column) by the same factor, is the same thing as to multiply the whole determinant by that factor.*

For in every element we have one constituent from any one particular row. Hence, if every constituent of that row is multiplied by the same factor, each element, and $\therefore$ the whole expression, is multiplied by the same factor.

The same is, of course, true for a column, instead of a row. For examples, see Art. 313.

324. PROP. 4. *If each constituent of a row (or column) is resolvable into the sum of two parts, the determinant is resolvable into two determinants.*

For suppose each constituent of the pth row is resolvable into the sum of two parts, thus suppose

$$a_{p1}=x_{p1}+y_{p1},$$
$$a_{pq}=x_{pq}+y_{pq},$$

then any element in which a_{pq} appears can be resolved into the sum of two parts differing from it only in having x_{pq} in one, and y_{pq} in the other, as a factor instead of a_{pq}; and since each

element contains a factor from the pth row, all the elements can be similarly resolved;

$$\therefore\ \Sigma\pm a_{11}\, a_{22}\ldots a_{pp}\ldots a_{nn}$$
$$=\Sigma\pm a_{11}a_{22}\ldots x_{pp}\ldots a_{nn}+\Sigma\pm a_{11}\, a_{22}\ldots y_{pp}\ldots a_{nn}.$$

The same is true for columns.

COR. It is evident by proceeding in the same way, that, if each constituent of a row (or column) is resolvable into the sum of any number of parts, the determinant is resolvable into the same number of determinants differing from the original determinant only in having in each one part, instead of the whole constituent, in that row or column.

325. By means of these propositions we can often reduce a determinant to an equivalent with smaller constituents.

Example. $$\begin{vmatrix} 3, & 1, & 2, & 3 \\ 4, & 0, & 2, & 1 \\ 6, & 4, & 1, & 2 \\ 7, & 3, & 0, & 1 \end{vmatrix} = \begin{vmatrix} 2+1, & 1, & 2, & 3 \\ 4+0, & 0, & 2, & 1 \\ 2+4, & 4, & 1, & 2 \\ 4+3, & 3, & 0, & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 1 & 2 & 3 \\ 4 & 0 & 2 & 1 \\ 2 & 4 & 1 & 2 \\ 4 & 3 & 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 2 & 3 \\ 0 & 0 & 2 & 1 \\ 4 & 4 & 1 & 2 \\ 3 & 3 & 0 & 1 \end{vmatrix}.$$

Now the second of these determinants has two columns identical, and therefore vanishes. We thus see that *we can subtract one row* (or column) *from another, without altering the value of the determinant.*

The first of these determinants has two as a factor of each constituent of its first column, and

$$\therefore,\ \text{Art. 323},\ =2\begin{vmatrix} 1 & 1 & 2 & 3 \\ 2 & 0 & 2 & 1 \\ 1 & 4 & 1 & 2 \\ 2 & 3 & 0 & 1 \end{vmatrix}.$$

EXAMPLES.—LXV.

1. Show that $\begin{vmatrix} 5 & 4 & 3 & 6 \\ 3 & 0 & 15 & 4 \\ 2 & 1 & 18 & 2 \\ 5 & 3 & 6 & 6 \end{vmatrix} = 6 \begin{vmatrix} 1 & 4 & 1 & 3 \\ 3 & 0 & 5 & 2 \\ 1 & 1 & 6 & 1 \\ 2 & 3 & 2 & 3 \end{vmatrix}$.

2. Show that $\begin{vmatrix} 1 & 6 & 3 & 4 \\ 3 & 6 & 0 & 1 \\ 7 & 3 & 7 & 2 \\ 2 & 0 & 6 & 2 \end{vmatrix} = 6 \begin{vmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 0 & 1 \\ 5 & 1 & 1 & 0 \\ 1 & 0 & 3 & 1 \end{vmatrix} = 6 \begin{vmatrix} 0 & 2 & 0 & 3 \\ 3 & 2 & 0 & 1 \\ 5 & 1 & 1 & 0 \\ 1 & 0 & 3 & 1 \end{vmatrix}$.

3. Show that $\begin{vmatrix} 5 & 2 & 2 & 1 \\ 4 & 2 & 3 & 3 \\ 6 & 3 & 7 & 2 \\ 3 & 1 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 & 1 \\ 0 & 2 & 0 & 3 \\ 0 & 3 & 5 & 2 \\ 1 & 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 3 \\ 0 & 3 & 5 & 2 \\ 1 & 1 & 1 & 1 \end{vmatrix}$.

4. Show that

$$\begin{vmatrix} 2 & 2 & 0 & 3 \\ 1 & 2 & 4 & 1 \\ 3 & 3 & 2 & 1 \\ 4 & 6 & 2 & 0 \end{vmatrix} = 2 \begin{vmatrix} 0 & 2 & 0 & 3 \\ 0 & -1 & 3 & 1 \\ 0 & 3 & 2 & 1 \\ -1 & 3 & 1 & 0 \end{vmatrix} = 2 \begin{vmatrix} 0 & 2 & 0 & 3 \\ 0 & -1 & 3 & 1 \\ 0 & 4 & -1 & 0 \\ -1 & 3 & 1 & 0 \end{vmatrix}.$$

326. Prop. 5. *In the determinant* $\Sigma \pm (1, 1)(2, 2) \ldots (n, n)$, *the coefficient of* (p, q) *is* $(-1)^{p-q}A_{p,q}$ (see Art. 320).

1° In each element in which (p, q) occurs, its coefficient is one of the products of $n-1$ constituents, of which one comes from each row and each column, with the exception of the pth row and qth column, and may therefore be obtained from the coefficient of (p, q) in any other element containing (p, q), by interchanges of second suffixes, taking care to change the sign of the product once for every interchange, according to Art. 314. The whole coefficient required is the sum of all such products.

2° We must then obtain some one element in which (p, q) occurs. Let $p>q$. If, starting from the principal element, we make successive interchanges amongst the second suffixes, thus q and $q+1$, then q and $q+2$, and so on, till we have interchanged q with p, we obtain the product

$$(-1)^{p-q}(1,1)\ldots(q-1,\,q-1)(q,\,q+1)$$
$$\ldots(p-1,p)(p,q)(p+1,p+1)\ldots(n,n).$$

3° Now in this, the coefficient of (p, q) is the principal element of $A_{p,q}$ multiplied by $(-1)^{p-q}$.

Hence by 1° the coefficient required is $(-1)^{p-q}$ multiplied by the sum of all the products, which can be obtained from the principal element of $A_{p,q}$ by interchanges of the second suffixes, prefixing $+$ or $-$ to each product, according as, to obtain it, we have had to make an even or odd number of interchanges.

But this sum is $A_{p,q}$ (Art. 314); $\therefore$ the required coefficient of (p, q) is $(-1)^{p-q}A_{p,q}$.

Similarly, if $p<q$, we can show that the coefficient of (p, q) is $(-1)^{q-p}A_{p,q}=(-1)^{p-q}A_{p,q}$.

COR. $\Sigma\pm(1,1)(2,2)\ldots(n,n)$

$$=(1,1)A_{1,1}-(1,2)A_{1,2}\ldots+(-1)^{n-1}(1,n)A_{1,n},$$

also $=(1,1)A_{1,1}-(2,1)A_{2,1}\ldots+(-1)^{n-1}(n,1)A_{n,1}$.

327. *Ex.* 1. Continuing the reduction of the determinant in Art. 325, we have

$$2\left\{\begin{vmatrix}0, & 2, & 1\\4, & 1, & 2\\3, & 0, & 1\end{vmatrix}-2\begin{vmatrix}1, & 2, & 3\\4, & 1, & 2\\3, & 0, & 1\end{vmatrix}+\begin{vmatrix}1, & 2, & 3\\0, & 2, & 1\\3, & 0, & 1\end{vmatrix}-2\begin{vmatrix}1, & 2, & 3\\0, & 2, & 1\\4, & 1, & 2\end{vmatrix}\right\}$$

$$=2\begin{vmatrix}0, & 2, & 1\\1, & 1, & 1\\3, & 0, & 1\end{vmatrix}-4\begin{vmatrix}1, & 2, & 1\\4, & 1, & 1\\3, & 0, & 1\end{vmatrix}+2\begin{vmatrix}1, & 0, & 2\\0, & 2, & 1\\3, & 0, & 1\end{vmatrix}-4\begin{vmatrix}1, & 0, & 2\\0, & 2, & 1\\4, & 1, & 2\end{vmatrix},$$

subtracting the 3rd row from the 2nd in the last determinant,

$$=2\begin{vmatrix}0, & 2, & 1\\1, & 1, & 1\\3, & 0, & 1\end{vmatrix}-4\begin{vmatrix}0, & 2, & 1\\1, & 1, & 0\\2, & 0, & 1\end{vmatrix}+2\begin{vmatrix}1, & 0, & 2\\0, & 2, & 1\\3, & 0, & 1\end{vmatrix}-4\begin{vmatrix}1, & 0, & 2\\0, & 2, & 1\\3, & 1, & 0\end{vmatrix},$$

subtracting the 3rd column from the 1st, and the 3rd row from the 2nd in the second det., and the 1st row from the 3rd in the last,

$$=2\left\{-\begin{vmatrix}2, & 1\\0, & 1\end{vmatrix}+3\begin{vmatrix}2, & 1\\1, & 1\end{vmatrix}\right\}-4\left\{-\begin{vmatrix}2, & 1\\0, & 1\end{vmatrix}+2\begin{vmatrix}2, & 1\\1, & 0\end{vmatrix}\right\}$$

$$+2\left\{\begin{vmatrix}2, & 1\\0, & 1\end{vmatrix}+3\begin{vmatrix}0, & 2\\2, & 1\end{vmatrix}\right\}-4\left\{\begin{vmatrix}2, & 1\\1, & 0\end{vmatrix}+3\begin{vmatrix}0, & 2\\2, & 1\end{vmatrix}\right\}$$

$$=2\{-2+3(2-1)\}-4\{-2-2\}+2\{2-12\}-4\{-1-12\}=50.$$

Ex. 2.

$$\begin{vmatrix}a^3 & b^3 & c^3\\a & b & c\\1 & 1 & 1\end{vmatrix}=\begin{vmatrix}a^3-b^3 & b^3 & c^3-b^3\\a-b & b & c-b\\0 & 1 & 0\end{vmatrix}$$

$$=(a-b)(c-b)\begin{vmatrix}a^2+ab+b^2 & b^3 & c^2+cb+b^2\\1 & b & 1\\0 & 1 & 0\end{vmatrix}$$

$$=(a-b)(c-b)\begin{vmatrix}a^2+ab-c^2-cb & b^3 & c^2+cb+b^2\\0 & b & 1\\0 & 1 & 0\end{vmatrix}$$

$$=(a-b)(c-b)(a-c)\begin{vmatrix}a+c+b & b^3 & c^2+cb+b^3\\0 & b & 1\\0 & 1 & 0\end{vmatrix}$$

$$=(a-b)(c-b)(a-c)(a+c+b)\begin{vmatrix}b & 1\\1 & 0\end{vmatrix}$$

$$=(a-b)(c-b)(a-c)(a+c+b)(-1)$$

$$=(a-b)(b-c)(c-a)(a+b+c).$$

It was evident at starting that $a-b$ is a factor of the determinant. For put $a=b$, and we get a determinant with two columns identical, and $\therefore\ =0$. Similarly for $b-c$ and $c-a$.

EXAMPLES.—LXVI.

1. Find the values of the determinants in *Ex.* LXV.

Calculate the values of the determinants,

2. $$\begin{vmatrix} 1, & 3, & 1, & 7 \\ 2, & 2, & 0, & 8 \\ 4, & 5, & 3, & 5 \\ 1, & 6, & 4, & 4 \end{vmatrix}.$$

3. $$\begin{vmatrix} 1, & 3, & 0, & 5 \\ 2, & 6, & 0, & 4 \\ 1, & 0, & 1, & 3 \\ 5, & 3, & 0, & 3 \end{vmatrix}.$$

Expand the following:—

4. $$\begin{vmatrix} u, & w', & v', & l \\ w', & v, & u', & m \\ v', & u', & w, & n \\ l, & m, & n, & 0 \end{vmatrix}.$$

5. $$\begin{vmatrix} 0, & ab^2, & ac^2 \\ ba^2, & 0, & bc^2 \\ ca^2, & cb^2, & 0 \end{vmatrix}.$$

6. Prove the following equation,

$$\frac{\begin{vmatrix} a & \lambda & \lambda \\ \lambda & b & \lambda \\ \lambda & \lambda & c \end{vmatrix}}{(a-\lambda)(b-\lambda)(c-\lambda)}=1+\lambda\left(\frac{1}{a-\lambda}+\frac{1}{b-\lambda}+\frac{1}{c-\lambda}\right).$$

7. Prove that

$$\begin{vmatrix} m+n-y+z, & -y+z-l, & -y+z-l \\ -z+x-m, & n+l-z+x, & -z+x-m \\ -x+y-n, & -x+y-n, & l+m-x+y \end{vmatrix}=0.$$

8. Show that if in any determinant, which is a function of x, n columns become the same when x is put equal to a, then the determinant is divisible by $(x-a)^{n-1}$.

9. Prove that

$$\begin{vmatrix} \beta+\gamma, & \gamma+\alpha, & \alpha+\beta, \\ \beta'+\gamma', & \gamma'+\alpha', & \alpha'+\beta', \\ \beta''+\gamma'', & \gamma''+\alpha'', & \alpha''+\beta'', \end{vmatrix}=2\begin{vmatrix} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{vmatrix}.$$

10. Find the value of the determinant

$$\begin{vmatrix} 1 & a & a^2 & a^3+bcd \\ 1 & b & b^2 & b^3+acd \\ 1 & c & c^2 & c^3+abd \\ 1 & d & d^2 & d^3+abc \end{vmatrix}.$$

11. If ω be an imaginary cube root of unity, what is the value of

$$\begin{vmatrix} 0, & 1, & \omega, & \omega^2 \\ -1, & 0, & -\omega^2, & \omega \\ -\omega, & \omega^2, & 0, & -1 \\ -\omega^2, & \omega, & 1, & 0 \end{vmatrix}?$$

12. Show that the determinant

$$\begin{vmatrix} 0, & a_1-a_2, & a_1-a_3, & \text{etc.}, & a_1-a_n \\ a_2-a_1, & 0, & a_2-a_3, & \text{etc.}, & a_2-a_n \\ a_3-a_1, & a_3-a_2, & 0, & \text{etc.}, & a_3-a_n \\ \text{etc.}, & \text{etc.}, & \text{etc.}, & & \\ a_n-a_1, & a_n-a_2, & \text{etc.}, & \text{etc.}, & 0 \end{vmatrix}$$

is zero, except when $n=2$.

13. Prove that

$$\begin{vmatrix} 0 & a & b & c \\ a & 0 & \gamma & \beta \\ b & \gamma & 0 & \alpha \\ c & \beta & \alpha & 0 \end{vmatrix} = (a\alpha+b\beta-c\gamma)^2-4ab\alpha\beta.$$

14. Prove that

$$\begin{vmatrix} 0, & 1, & 1, & 1 \\ 1, & 0, & z^2, & y^2 \\ 1, & z^2, & 0, & x^2 \\ 1, & y^2, & x^2, & 0 \end{vmatrix} = \frac{1}{x^2y^2z^2} \begin{vmatrix} 0, & x, & y, & z \\ x, & 0, & xyz^2, & xy^2z \\ y, & xyz^2, & 0, & x^2yz \\ z, & xy^2z, & x^2yz, & 0 \end{vmatrix}.$$

15. Show that $\begin{vmatrix} 1, & \gamma, & -\beta \\ -\gamma, & 1, & \alpha \\ \beta, & -\alpha, & 1 \end{vmatrix} = 1+\alpha^2+\beta^2+\gamma^2.$

328. PROP. 6. *To show that the product*

$\Sigma \pm a_{11}\, a_{22} \ldots a_{nn} \times \Sigma \pm b_{11}\, b_{22} \ldots b_{nn}$ *is equal to*

$$\begin{vmatrix} a_{11}b_{11}+a_{12}b_{12}+a_{13}b_{13}+\&c., & a_{11}b_{21}+a_{12}b_{22}+\&c., & \ldots & a_{11}b_{n1}+a_{12}b_{n2}+\&c. \\ a_{21}b_{11}+a_{22}b_{12}+a_{23}b_{13}+\&c., & a_{21}b_{21}+a_{22}b_{22}+\&c., & \ldots & a_{21}b_{n1}+a_{22}b_{n2}+\&c. \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1}b_{11}+a_{n2}b_{12}+a_{n3}b_{13}+\&c., & a_{n1}b_{21}+a_{n2}b_{22}+\&c., & \ldots & a_{n1}b_{n1}+a_{n2}b_{n2}+\&c. \end{vmatrix}$$

Call these determinants A, B, C respectively.

We shall use the term *partial column* to denote any one of the n columns into which each of the columns of C is divided (Art. 324). Thus the rth partial column of the pth column of C is

$$\begin{matrix} a_{1r}\, b_{pr} \\ a_{2r}\, b_{pr} \\ \vdots \\ a_{nr}\, b_{pr} \end{matrix}$$

Thus C is the sum of the n^n partial determinants, each of which is formed by taking a partial column from each of C's columns (Art. 324); and b_{pr} being a factor of every constituent in the above column, we should write b_{pr} outside any partial determinant in which it occurred, leaving the column

$\begin{matrix} a_{1r} \\ a_{2r} \\ \vdots \\ a_{nr} \end{matrix}$ inside, as the rth column.

It will be observed that this rth column is the same, from whichever of C's columns it is taken. Hence, if in forming any partial determinant, we took the rth column from two of C's columns, we should obtain two columns identical, and therefore that partial determinant would vanish.

Therefore all the partial determinants, which do not vanish, must be made up by taking a partial column of a different position from each of C's columns. Hence each of these determinants must be of the form

$$b_{1\alpha} b_{2\beta} b_{3\gamma} \dots b_{n\nu} \begin{vmatrix} a_{1\alpha}, a_{1\beta}, \dots a_{1\nu} \\ a_{2\alpha}, a_{2\beta}, \dots a_{2\nu} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_{n\alpha}, a_{n\beta}, \dots a_{n\nu} \end{vmatrix} \quad . \quad . \quad . \quad I,$$

where α, β, γ, . . . ν is some arrangement of the numbers 1, 2, . . . n.

In this expression the determinant is merely A with its columns arranged differently, and $\therefore = \pm A$.

Also, suppose the arrangement α, β, γ, . . . ν to be obtainable from 1, 2, 3, . . . n by x interchanges, then the above determinant $=(-1)^x A$ (Art. 322, Cor. I),

$$\text{and } I = A(-1)^x b_{1\alpha} b_{2\beta} \dots b_{n\nu}.$$

So that A is a factor of each of these terms, of which C is the sum;

$$\therefore\ C = A\Sigma\{(-1)^x b_{1\alpha} b_{2\beta} \dots b_{n\nu}\}, \text{ which is } A.B,$$

since the product $b_{1\alpha} b_{2\beta}$ . . . $b_{n\nu}$ is obtainable, from the principal element of B, by x interchanges amongst the second suffixes.

The student will observe that C can be written down by the following rule :—

The kth row of the lth column of C is formed by multiplying together the kth and lth rows of A and B respectively, constituent by constituent.

EXAMPLES.—LXVII.

1. Write down the products of the following pairs of determinants :—

(1) $\begin{vmatrix} a, & b \\ c, & d \end{vmatrix} \times \begin{vmatrix} \alpha, & \beta \\ \gamma, & \delta \end{vmatrix}.$

(2) $\begin{vmatrix} a & b & c \\ \alpha & \beta & \gamma \\ x & y & z \end{vmatrix} \times \begin{vmatrix} a' & b' & c' \\ \alpha & \beta & \gamma \\ x & y & z \end{vmatrix}.$

2. Express $\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}^2$ in the form of a determinant.

3. Show that

$$\begin{vmatrix} a^2+b^2+c^2, & a\alpha+b\beta+c\gamma \\ a\alpha+b\beta+c\gamma, & \alpha^2+\beta^2+\gamma^2 \end{vmatrix} = \begin{vmatrix} b, & c \\ \beta, & \gamma \end{vmatrix}^2 + \begin{vmatrix} c, & a \\ \gamma, & \alpha \end{vmatrix}^2 + \begin{vmatrix} a, & b \\ \alpha, & \beta \end{vmatrix}^2.$$

4. Express $\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \times \begin{vmatrix} a & b \\ \alpha & \beta \end{vmatrix}$ in the form of a determinant.

5. If in $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ A_2 be the minor of a_2, B_3 the minor of b_3, and so on,

prove that $\dfrac{1}{a_1}\begin{vmatrix} B_2, & B_3 \\ C_2, & C_3 \end{vmatrix} =$ the original determinant.

6. The square of a determinant is a symmetrical determinant.

329. We will now give an instance of the use of determinants in the solution of equations, and in elimination.

To solve the n equations,

$$a_1x+b_1y+c_1z+\text{ etc. } \ldots = m_1, \quad (1),$$
$$a_2x+b_2y+\text{ etc. } \ldots\ldots = m_2, \quad (2),$$
$$\text{etc.} \qquad = \text{etc.}$$
$$a_nx+b_ny+\text{ etc. } \ldots\ldots = m_n, \quad (n).$$

Consider the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 & \ldots \\ a_2 & b_2 & \ldots & \ldots \\ \text{etc.} & & \text{etc.} & \\ a_n & b_n & \ldots & \ldots \end{vmatrix},$$

denote it by D.

Multiply (1) by the minor of a_1, (2) by the minor of a_2, with the negative sign prefixed, and, generally, the equation (h) by the minor of a_h with the symbol $(-1)^{h-1}$ prefixed, and then add all the results together. The coefficient of x will be D (Art. 326).

The coefficient of y will be what D becomes when b_1, b_2, etc. are written for a_1, a_2, etc., *i.e.*, it will be a determinant with two identical columns, and therefore will vanish. For similar reasons the coefficients of z, etc. all vanish.

Also on the right hand we have what D becomes when m_1, m_2, etc. are written for a_1, a_2, etc., *i.e.*, the determinant $\Sigma\pm m_1b_2c_3 \ldots$;

$$\therefore\ x=\frac{\Sigma\pm m_1b_2c_3\ \ldots}{\Sigma\pm a_1b_2\ \ldots\ .}.$$

Similarly $y=\dfrac{\Sigma\pm a_1m_2c_3\ \ldots}{\Sigma\pm a_1b_2\ \ldots\ .}$; $z=$ etc.; etc.

COR. If $m_1=m_2=$ etc. $=0$, we can eliminate x, y, etc., and the result is $\Sigma\pm a_1b_2 \ldots =0$.

EXAMPLES.—LXVIII.

Solve, by using determinants, the equations :—

1. $$\begin{aligned} 2x+y+3z&=2\\ x+2z&=4\\ 3y+z&=5. \end{aligned}$$

2. $$\begin{aligned} x+y+z&=0\\ (b+c)x+(c+a)y+(a+b)z&=0\\ bcx+cay+abz&=1. \end{aligned}$$

3. $$\begin{aligned} x+2y+3u&=5\\ -y+z+u&=3\\ y+2z-u+2x&=4\\ x+z-3u&=1. \end{aligned}$$

4. $$\begin{aligned} 2x+3y+z&=3\\ x+z+u&=0\\ y+2z+3u&=4\\ 3z-2u&=-2. \end{aligned}$$

5. Eliminate x, y, z from the equations

$$\begin{aligned} ax+hy+fz-l&=0\\ hx+by+ez-m&=0\\ fx+ey+cz-n&=0\\ lx+my+nz&=0. \end{aligned}$$

6. Eliminate x, y, z from the equations

$$\frac{a^2}{x^2}+\frac{b^2}{y^2}+\frac{c^2}{z^2}=0$$

$$\frac{al}{x^2}+\frac{bm}{y^2}+\frac{cn}{z^2}=0$$

$$\frac{al'}{x^2}+\frac{bm'}{y^2}+\frac{cn'}{z^2}=0.$$

7. If $L=ax+by+dz$
$M=bx+cy+ez$
$N=dx+ey+fz$ and $-H=\begin{vmatrix} a & b & d \\ b & c & e \\ d & e & f \end{vmatrix}$,
$P=lx+my+nz$,

rove that $\begin{vmatrix} 0 & l & m & n \\ L & a & b & d \\ M & b & c & e \\ N & d & e & f \end{vmatrix}=P.H.$

8. Prove that $\begin{vmatrix} (a+b)^2, & bc, & ac \\ bc, & (a+c)^2, & ab \\ ac, & ab, & (b+c)^2 \end{vmatrix}=2(a+b+c)abc.$

9. Show that $(x+y+z)(x-y-z)(y-z-x)(z-x-y)$

$$=\begin{vmatrix} 0 & x & y & z \\ x & 0 & z & y \\ y & z & 0 & x \\ z & y & x & 0 \end{vmatrix}.$$

10. Prove that $\begin{vmatrix} x'x'', & y'y'', & z'z'' \\ x''x, & y''y, & z''z \\ x\,x', & y\,y', & z\,z' \end{vmatrix}=\begin{vmatrix} y\,z, & z\,x, & x\,y, \\ y'z', & z'x', & x'y', \\ y''z'', & z''x'', & x''y'' \end{vmatrix}.$

XXIV

Finite Differences.

330. In this Chapter we shall discuss some problems leading to what really are particular cases of equations in a branch of higher Mathematics called *Finite Differences*.

The methods therefore will be best explained by giving several *examples*.

331. *Ex.* 1. Obtain the nth term of the recurring series,

$$u_0 + u_1 x + u_2 x^2 + \text{ etc.},$$

the scale of relation being $1 - px - qx^2 = 0$.

We have, then, to find such a form for the function u_n as will satisfy the relation, $u_n - pu_{n-1} - qu_{n-2} = 0$. . . (1).

Now $A\alpha^n$ is such a form, when A is *any* constant, and α a root of the equation, $\alpha^2 - p\alpha - q = 0$. . . . (2).

This the student will see at once by substituting, in (1), $A\alpha^n$ for u_n, $A\alpha^{n-1}$ for u_{n-1}, $A\alpha^{n-2}$ for u_{n-2}.

Let α, β be the roots of (2), and A, B two constants (*i.e.* symbols which do not change when n is changed from one value to another, differing from the former by an integer); then not only $A\alpha^n$ and $B\beta^n$, but $A\alpha^n + B\beta^n$ are forms of u_n which satisfy (1). The latter form is the most general solution of (1), and therefore the most general form which can be found for the nth term of *any* series, in which the scale of relation is that given above.

We have still to determine A and B, so that it may be the nth term of the particular given series.

Putting $n=0$, and 1, successively, we have

$$A + B = u_0$$

$$A\alpha + B\beta = u_1;$$

two equations which determine A and B.

332. We have stated that $A\alpha^n+B\beta^n$ is the most general solution of (1). This we cannot prove here, but the student will find that no other functions, except such as are particular cases of the above, will satisfy (1), if α and β are different. If, however, $4q=-p^2$, $\alpha=\beta=\frac{p}{2}$. In this case, the general form of u_n is $(C+Dn)\alpha^n$, C and D being constants.

This the student can easily verify for himself by substituting, in (1), $(C+Dn)\alpha^n$ for u_n, $(C+D\overline{n-1})\alpha^{n-1}$ for u_{n-1}, $(C+D\overline{n-2})\alpha^{n-2}$ for u_{n-2}, and remembering that $q=-\frac{p^2}{4}$, $\alpha=\frac{p}{2}$.

Then C and D can be determined from the equations,

$$C=u_0, \quad \text{putting } n=0,$$
$$\text{and } (C+D)\alpha=u_1 \quad \text{,,} \quad n=1.$$

If the scale of relation were $1-px-qx^2-rx^3=0$, and α, β, γ the roots of $a^3-qa^2-pa-r=0$, the general form of u_n would be

$$A\alpha^n+B\beta^n+C\gamma^n, \text{ when } \alpha, \beta, \gamma \text{ are all different,}$$
$$A\alpha^n+(B+Cn)\beta^n, \quad \text{,,} \quad \beta=\gamma, \text{ and } \alpha \text{ is different,}$$
$$(A+Bn+Cn^2)\alpha^n, \quad \text{,,} \quad \alpha=\beta=\gamma;$$

A, B, C in each case being constants, such as we had in Art. 331, and so on, for other scales of relation.

EXAMPLES.—LXIX.

Determine the general term of each of the following recurring series :—

1. $1+3x+4x^2+7x^3+11x^4+18x^5+$ etc.
2. $3+6x+14x^2+36x^3+98x^4+276x^5+$ etc.
3. $3+11x+31x^2+95x^3+283x^4+$ etc.
4. $3+15x+63x^2+243x^3+$ etc.

5. $2+4x+6x^2-54x^4-$ etc.

6. $1+2x+3x^2+5x^3+7x^4+9x^5+$ etc.

7. Show that the $(r+1)$th term of the recurring series, $2-5+29-89+$ etc., is

$$\tfrac{1}{7}\{3^{r+1}+11.2^{2r}(-1)^r\}.$$

8. If u_x be the coefficient of t^x in the development of $\dfrac{1-2t-2t^3}{1+t+t^2+t^3}$, show that $u_{x+3}+u_{x+2}+u_{x+1}+u_x=0$; and hence find u_x.

9. A recurring series is such that each term is the sum of the two preceding. Find the limiting ratio of the nth to the $(n-1)$th term, when n is infinite.

333. *Ex.* 2. Find the nth convergent of the continued fraction,

$$\frac{a}{a+}\,\frac{a}{a+}\,\frac{a}{a+}\text{ etc.}$$

Let $\dfrac{p_n}{q_n}$ denote the nth convergent; then the form for p_n must satisfy the equation $p_n=ap_{n-1}+ap_{n-2}$, . . . (1).

Hence, as in Art. 331, if h, k be the roots of the equation, $x^2=ax+a$, Ah^n+Bk^n is the most general form for p_n; and also for q_n, since the function for q_n must satisfy the equation $q_n=aq_{n-1}+aq_{n-2}$, which is the same in form as (1).

To determine A *and* B. The first two convergents are

$$\frac{a}{a},\ \frac{aa}{a^2+a}.$$

Hence, *considering numerators*, and putting $n=1$, and 2, successively, we have

$$Ah+Bk=a,$$
$$Ah^2+Bk^2=aa,$$

two equations from which the values of A and B for p_n can be obtained.

Similarly, *considering denominators*, we have

$$Ah + Bk = a$$
$$Ah^2 + Bk^2 = a^2 + a.$$

Obs. If $a = -\frac{a^2}{4}$, $h = k$, and the general form for p_n, and for q_n, is $(C+Dn)h^n$, and C and D can be determined from the equations,

$$\left.\begin{array}{l}(C+D)h = a \\ (C+2D)h^2 = \alpha a\end{array}\right\} \text{for numerators,}$$

$$\text{and } \left.\begin{array}{l}(C+D)h = a \\ (C+2D)h^2 = a^2 + \alpha\end{array}\right\} \text{for denominators.}$$

EXAMPLES.—LXX.

1. Find the value to n terms of the continued fraction,

$$\frac{2}{1+}\,\frac{2}{1+}\,\frac{2}{1+} \text{ etc.}$$

2. Find the nth convergent of,

$$(1)\ \frac{4}{5-}\,\frac{4}{5-}\,\frac{4}{5-} \text{ etc.,}$$

$$(2)\ \frac{5}{4+}\,\frac{5}{4+}\,\frac{5}{4+} \text{ etc.,}$$

$$(3)\ \frac{5}{5-}\,\frac{5}{5-}\,\frac{5}{5-} \text{ etc.}$$

3. The nth convergent to the continued fraction,

$$4+\frac{4}{8+}\,\frac{4}{8+}\,\frac{4}{8+} \cdots,$$

is twice the nth convergent to the continued fraction,

$$2+\frac{1}{4+}\,\frac{1}{4+}\,\frac{1}{4+} \cdots$$

4. Show that $\frac{n}{n+1} = \frac{1}{2-}\,\frac{1}{2-}\,\frac{1}{2} \cdots$, there being n quotients of 2.

5. If $\frac{p_n}{q_n}$ be the nth convergent to $\sqrt{a^2+1}$, prove that

$$\frac{p_n}{q_n}=\sqrt{a^2+1}\,\frac{(a+\sqrt{a^2+1})^n+(a-\sqrt{a^2+1})^n}{(a+\sqrt{a^2+1})^n-(a-\sqrt{a^2+1})^n}.$$

334. *Ex.* 3. Find $\frac{p_n}{q_n}$, the nth convergent of

$$\frac{3}{1+}\,\frac{4}{1+}\,\frac{3}{1+}\,\frac{4}{1+}\text{ etc.}$$

First, let n be odd and $=2m+1$;

$$\therefore\ p_{2m+1}=p_{2m}+3p_{2m-1},$$
$$p_{2m}=p_{2m-1}+4p_{2m-2},$$
$$p_{2m-1}=p_{2m-2}+3p_{2m-3}.$$

Eliminating p_{2m}, p_{2m-2}, *i.e.*, the numerators of even rank, we have $$p_{2m+1}=8p_{2m-1}-12p_{2m-3}.$$

Put $p_{2m+1}=Ax^{2m+1}$, where A is constant, as in Art. 331; then $x^4=8x^2-12$.

The roots of this equation are $\pm\sqrt{6}$, $\pm\sqrt{2}$; $\therefore$ the general form for p_{2m+1} is

$$A(\sqrt{6})^{2m+1}+B(-\sqrt{6})^{2m+1}+C(\sqrt{2})^{2m+1}+D(-\sqrt{2})^{2m+1}$$
$$=(A-B)6^{\frac{2m+1}{2}}+(C-D)2^{\frac{2m+1}{2}}.$$

Now $p_1=3$; $\qquad\therefore\ (A-B)\sqrt{6}+(C-D)\sqrt{2}=3$;

and $p_3=12$; $\qquad\therefore\ 6(A-B)\sqrt{6}+2(C-D)\sqrt{2}=12$;

$$\therefore\ (A-B)\sqrt{6}=\frac{3}{2},\ (C-D)\sqrt{2}=\frac{3}{2};$$

$$\therefore\ p_{2m+1}=\frac{3}{2}(6^m+2^m)=3.2^{m-1}(3^m+1);$$

$$\therefore\ p_{2m-1}=3.2^{m-2}(3^{m-1}+1).$$

Also $$p_{2m}=p_{2m+1}-3p_{2m-1}$$
$$=3.2^{m-2}(3^m-1).$$

Again, the general form for q_n being the same as for p_n, we have for denominators, since $q_1=1$, $q_3=8$,

$$(A-B)\sqrt{6}+(C-D)\sqrt{2}=1,$$
$$6(A-B)\sqrt{6}+2(C-D)\sqrt{2}=8;$$
$$\therefore\ (A-B)\sqrt{6}=\frac{3}{2},\quad (C-D)\sqrt{2}=-\frac{1}{2}.$$

Whence we find, in the same way as for p_{2m+1} and p_{2m-1},

$$q_{2m+1}=2^{m-1}(3^{m+1}-1),$$
$$q_{2m}=2^{m-2}(3^{m+1}+1).$$

Therefore $$\frac{p_{2m+1}}{q_{2m+1}}=\frac{3.2^{m-1}(3^m+1)}{2^{m-1}(3^{m+1}-1)}=3.\frac{3^m+1}{3^{m+1}-1},$$

and $$\frac{p_{2m}}{q_{2m}}=\frac{3.2^{m-2}(3^m-1)}{2^{m-2}(3^{m+1}+1)}=3.\frac{3^m-1}{{}^{+1}+1}.$$

Hence, if n be odd, $$\frac{p_n}{q_n}=3.\frac{3^{\frac{n-1}{2}}+1}{3^{\frac{n+1}{2}}-1}$$

and, if n be even, $$\frac{p_n}{q_n}=3.\frac{3^{\frac{n}{2}}-1}{3^{\frac{n}{2}+1}+1}.$$

335. *Ex.* 4. Find the nth convergent of the continued fraction

$$\frac{1}{3-}\ \frac{4}{3-}\ \frac{1}{3-}\ \frac{4}{3-}\ \text{etc.}$$

First, let n be odd and $=2m+1$;

$$\therefore\ p_{2m+1}=3p_{2m}-p_{2m-1},$$
$$p_{2m}=3p_{2m-1}-4p_{2m-2},$$
$$p_{2m-1}=3p_{2m-2}-p_{2m-3}.$$

Eliminating p_{2m} and p_{2m-2}, we have

$$p_{2m+1}=4p_{2m-1}-4p_{2m-3}.$$

Put $p_{2m+1}=Ax^{2m+1}$, A being a constant;

$$\therefore\ x^4=4x^2-4.$$

This equation has two roots each equal to $\sqrt{2}$, and two others each equal to $-\sqrt{2}$;

$$\therefore\ p_{2m+1}=\{A+B(2m+1)\}2^{\frac{2m+1}{2}}-\{C+D(2m+1)\}2^{\frac{2m+1}{2}}$$

$$=(A-C+B-D)2^{\frac{2m+1}{2}}+2m(B-D)2^{\frac{2m+1}{2}}.$$

Now $p_1=1;\quad \therefore\ (A-C+B-D)\sqrt{2}=1,$

$p_3=8;\quad \therefore\ 2(A-C+B-D)\sqrt{2}+4(B-D)\sqrt{2}=8;$

$$\therefore\ A-C+B-D=\frac{1}{\sqrt{2}},\ B-D=\frac{3}{2\sqrt{2}};$$

$$\therefore\ p_{2m+1}=2^m+3m2^m.$$

Also $p_{2m}=\frac{1}{3}(p_{2m+1}+p_{2m-1})=3m2^{m-1}.$

Also, the general form for q_n being the same as for p_n, we have for denominators, since $q_1=3$, $q_3=12$,

$$(A-C+B-D)\sqrt{2}=3,$$

$$2(A-C+B-D)\sqrt{2}+4(B-D)\sqrt{2}=12;$$

$$\therefore\ (A-C+B-D)=\frac{3}{\sqrt{2}},\ (B-D)=\frac{3}{2\sqrt{2}};$$

$$\therefore\ q_{2m+1}=3.2^m+3m2^m=3(m+1)2^m,$$

and $q_{2m}=(3m+2)2^{m-1}.$

Therefore $\dfrac{p_{2m+1}}{q_{2m+1}}=\dfrac{2^m+3m2^m}{3(m+1)2^m}=\dfrac{1+3m}{3(m+1)},$

$$\frac{p_{2m}}{q_{2m}}=\frac{3m.2^{m-1}}{(3m+2)2^{m-1}}=\frac{3m}{3m+2}.$$

Hence, if n be odd, $\dfrac{p_n}{q_n}=\dfrac{3n-1}{3(n+1)},$

and, if n be even, $\dfrac{p_n}{q_n}=\dfrac{3n}{3n+4}.$

EXAMPLES.—LXXI.

Find the nth convergents of the following:—

1. $\dfrac{3}{2+}\,\dfrac{3}{3+}\,\dfrac{3}{2+}\,\dfrac{3}{3+}$ etc.

2. $\dfrac{2}{1-}\,\dfrac{2}{3+}\,\dfrac{2}{1-}\,\dfrac{2}{3+}$ etc.

3. $\dfrac{2}{1+}\,\dfrac{3}{2+}\,\dfrac{2}{1+}\,\dfrac{3}{2+}$ etc.

4. $\dfrac{9}{2+}\,\dfrac{1}{-2+}\,\dfrac{9}{2+}\,\dfrac{1}{-2+}$ etc.

5. $\dfrac{1}{1+}\,\dfrac{4}{-1+}\,\dfrac{1}{1+}\,\dfrac{4}{-1+}$ etc.

6. $\dfrac{a^2}{1+}\,\dfrac{a^2-4}{1+}\,\dfrac{a^2}{1+}\,\dfrac{a^2-4}{1+}$ etc.

336. In the following class of examples the principal difficulty is similar to that in Artt. **196, 199, 203**.

Ex. 5. Show that $\dfrac{1}{4-}\,\dfrac{3}{6-}\,\dfrac{5}{8-}$ etc. $\dfrac{2n-1}{2n+2}=\dfrac{3S+1}{3S+4}$,

$$\text{where } S\equiv\frac{2^2\,\underline{|2}}{\underline{|5}}+\ .\ .\ .\ +\frac{2^n\,\underline{|n}}{\underline{|2n+1}}.$$

Let $\dfrac{p_n}{q_n}$ denote the nth convergent, then the relation between p_n, p_{n-1}, p_{n-2} is

$$p_n=(2n+2)p_{n-1}-(2n-1)p_{n-2}\quad .\quad .\quad .\quad (1)$$

$$=(2n+1)p_{n-1}+p_{n-1}-(2n-1)p_{n-2};$$

$$\therefore\ p_n-(2n+1)p_{n-1}=p_{n-1}-(2n-1)p_{n-2},\quad .\quad (2).$$

That is, we have transformed (1) into an equivalent equation

(2), of which one side is the same function of n, that the other is of $n-1$;

$$\left.\begin{aligned} \therefore\ p_n-(2n+1)p_{n-1}&=A,\\ p_{n-1}-(2n-1)p_{n-2}&=A,\\ \text{etc.}\qquad &=\text{etc.}\\ p_2-5p_1\qquad &=A, \end{aligned}\right\} \quad . \quad . \quad (3),$$

where A is some constant, *i.e.* a symbol which does not change in value, when n is changed by an integer.

We shall now transform these into equivalent equations, in each of which the left-hand side will consist of the difference of two terms, which will be respectively the same functions of consecutive integers.

Divide the first by $1.3.7\ldots(2n-1)(2n+1)$, the second by $1.3.7\ldots(2n-1)$, etc., and the last by 1.3.5.

We have

$$\frac{p_n}{1.3.7\ldots(2n+1)}-\frac{p_{n-1}}{1.3.7\ldots(2n-1)}=\frac{A}{1.3.7\ldots(2n-1)(2n+1)},$$

$$\frac{p_{n-1}}{1.3.7\ldots(2n-1)}-\frac{p_{n-2}}{1.3.7\ldots(2n-3)}=\frac{A}{1.3.7\ldots(2n-1)},$$

$$\text{etc.}\quad - \quad \text{etc.} \quad = \quad \text{etc.}$$

$$\frac{p_2}{1.3.5}\quad - \quad \frac{p_1}{1.3} \quad = \quad \frac{A}{1.3.5};$$

$\therefore$, adding and remembering that $\dfrac{1}{1.3.7\ldots(2n+1)}=\dfrac{2^n\underline{|n}}{\underline{|2n+1}}$,

we have $\dfrac{2^n\underline{|n}\,p_n}{\underline{|2n+1}}-\dfrac{p_1}{3}=A\left\{\dfrac{2^2\underline{|2}}{\underline{|5}}+\ldots+\dfrac{2^{n-1}\underline{|n-1}}{\underline{|2n-1}}+\dfrac{2^n\underline{|n}}{\underline{|2n+1}}\right\}$

$$=AS.$$

Now $\qquad p_1=1,\ p_2=6;$

$\therefore$, from the last equation of (3), $A=6-5=1$;

$$\therefore\ \frac{p_n 2^n\underline{|n}}{\underline{|2n+1}}=\frac{1}{3}+S.$$

For denominators. Since the equation for q_n corresponding to (1) is of identically the same form, we have

$$\frac{q_n 2^n \underline{|n}}{\underline{|2n+1}} = \frac{q_1}{3}+AS,$$

$$\text{and } q_2-5q_1=A.$$

Now $q_2=21$, $q_1=4$;

$$\therefore\ A=21-20=1.$$

Therefore $\dfrac{q_n 2^n \underline{|n}}{\underline{|2n+1}}=\dfrac{4}{3}+S.$ Thus $\dfrac{p_n}{q_n}=\dfrac{1+3S}{4+3S}.$

337. *Ex.* 6. Find the nth convergent of the continued fraction

$$\frac{3}{-1+}\ \frac{8}{-1+}\ \frac{15}{-1+}\ \text{etc.}\ \frac{n^2-1}{-1+}\ \text{etc.}$$

Here $p_n=-p_{n-1}+(n^2-1)p_{n-2}$ . . . . . (1),

$$=np_{n-1}-(n+1)p_{n-1}+(n^2-1)p_{n-2};$$

$$\therefore\ p_n-np_{n-1}=-(n+1)\{p_{n-1}-\overline{n-1}p_{n-2}\};$$

$$\therefore\ (-1)^n p_n+n(-1)^{n-1}p_{n-1}=(n+1)\{(-1)^{n-1}p_{n-1} +(n-1)(-1)^{n-2}p_{n-2}\};$$

$$\therefore\ \frac{(-1)^n p_n}{\underline{|n+1}}+\frac{n(-1)^{n-1}p_{n-1}}{\underline{|n+1}}=\frac{(-1)^{n-1}p_{n-1}}{\underline{|n}} +\frac{(n-1)(-1)^{n-2}p_{n-2}}{\underline{|n}};$$

$$\therefore\ \frac{(-1)^n p_n}{\underline{|n+1}}+\frac{n(-1)^{n-1}p_{n-1}}{\underline{|n+1}}=A,$$

where A is a symbol which does not change with n.

Again $$\frac{p_n}{\underline{|n+1}}-\frac{np_{n-1}}{\underline{|n+1}}=A(-1)^n;$$

$$\therefore\ \frac{p_n}{\underline{|n}}-\frac{p_{n-1}}{\underline{|n-1}}=A(-1)^n(1+n).$$

Similarly $\dfrac{p_{n-1}}{\underline{|n-1}}-\dfrac{p_{n-2}}{\underline{|n-2}}=A(-1)^{n-1}n,$

$$\text{etc.} - \text{etc.} = \text{etc.},$$

$$\frac{p_2}{\underline{|2}}-\frac{p_1}{1}=A3\,;$$

$\therefore$ adding

$$\frac{p_n}{\underline{|n}}-p_1=A\{3-4+5-\text{etc.}+\ldots+(-1)^{n-1}n+(-1)^n(n+1)\}$$

$$\left.\begin{aligned} &=-A\frac{(n-1)}{2}, \text{ if } n \text{ be odd,}\\ \text{and} \quad &=\frac{A(n+4)}{2}, \text{ if } n \text{ be even.}\end{aligned}\right\} \quad . \quad . \quad . \quad (2).$$

Now $p_1=3$, $p_2=-3$; $\therefore$, putting $n=2$, we have

$$-\frac{3}{2}-3=A3\,; \quad \therefore A=-\frac{3}{2}\,;$$

$$\therefore \frac{p_n}{\underline{|n}}=3+\frac{3}{4}(n-1)=\frac{3(n+3)}{4}, \text{ if } n \text{ be odd,}$$

$$\text{and} \quad =3-\frac{3}{4}(n+4)=-\frac{3n}{4}, \text{ if } n \text{ be even.}$$

For denominators. Since the equation for q_n corresponding to (1) is of identically the same form, we have, as in (2),

$$\frac{q_n}{\underline{|n}}-q_1=-\frac{A(n-1)}{2}, \text{ or } \frac{A(n+4)}{2}, \text{ according as } n \text{ is odd, or even.}$$

Now $q_1=-1$, $q_2=9$; $\therefore$, putting $n=2$, we have

$$\therefore \frac{9}{\underline{|2}}+1=\frac{A}{2}(2+4)\,; \quad \therefore A=\frac{11}{6}\,;$$

$$\therefore \frac{q_n}{\underline{|n}}=-1-\frac{11}{12}(n-1)=-\frac{11n+1}{12}, \text{ if } n \text{ be odd,}$$

$$\text{and } =-1+\frac{11}{12}(n+4)=\frac{11n+32}{12}, \text{ if } n \text{ be even;}$$

$$\therefore \frac{p_n}{q_n}=-\frac{9(n+3)}{11n+1}, \text{ if } n \text{ be odd,}$$

$$\text{and } =-\frac{9n}{11n+32}, \text{ if } n \text{ be even.}$$

338. *Ex.* 7. Prove that $\frac{1}{2+}\,\frac{2}{3+}\,\frac{3}{4+}$ etc. $=\frac{3-e}{e-2}$.

If $\frac{p_n}{q_n}$ be the nth convergent, we have

$$p_n=(n+1)p_{n-1}+np_{n-2}, \quad . \quad . \quad . \quad . \qquad (1);$$

$$\therefore\ (n+2)p_n=(n+1)(n+2)p_{n-1}+(n+2)np_{n-2}$$
$$=(n+1)(n+3)p_{n-1}-(n+1)p_{n-1}+(n+2)np_{n-2};$$
$$\therefore\ (n+2)p_n-(n+1)(n+3)p_{n-1}=-(n+1)p_{n-1}+(n+2)np_{n-2};$$
$$\therefore\ (-1)^n(n+2)p_n+(-1)^{n-1}(n+1)(n+3)p_{n-1}$$
$$=(-1)^{n-1}(n+1)p_{n-1}+(-1)^{n-2}n(n+2)p_{n-2};$$
$$\therefore\ (-1)^n(n+2)p_n+(-1)^{n-1}(n+1)(n+3)p_{n-1}=A, \text{ a const. } (2);$$
$$\therefore\ (n+2)p_n-(n+1)(n+3)p_{n-1}=A(-1)^n;$$

$$\left.\begin{aligned}\therefore\ \frac{n+2}{|\underline{n+3}}p_n-\frac{n+1}{|\underline{n+2}}p_{n-1}&=\frac{A(-1)^n}{|\underline{n+3}},\\ \frac{n+1}{|\underline{n+2}}p_{n-1}-\frac{n}{|\underline{n+1}}p_{n-2}&=\frac{A(-1)^{n-1}}{|\underline{n+2}},\\ \text{etc.}\quad -\quad \text{etc.}\quad &=\quad \text{etc.}\\ \frac{4}{|\underline{5}}p_2-\frac{3}{|\underline{4}}p_1\qquad &=\frac{A}{|\underline{5}};\end{aligned}\right\} \quad . \quad . \qquad (3);$$

$$\therefore\ \frac{n+2}{|\underline{n+3}}p_n=\frac{3}{|\underline{4}}p_1+A\left\{\frac{1}{|\underline{5}}-\frac{1}{|\underline{6}}+\ .\ .\ .\ +\frac{(-1)^n}{|\underline{n+3}}\right\}.$$

Now $\qquad p_1=1,\ p_2=3;$

$\therefore$, from the last of equations (3),

$$\frac{4.3}{|\underline{5}}-\frac{3}{|\underline{4}}=\frac{A}{|\underline{5}};\ \therefore\ A=-3;$$

$$\therefore\ \frac{n+2}{|\underline{n+3}}p_n=\frac{3}{|\underline{4}}-3\left\{\frac{1}{|\underline{5}}-\frac{1}{|\underline{6}}+\ .\ .\ .\ +\frac{(-1)^n}{|\underline{n+3}}\right\}$$
$$=-1+3\left\{1-\frac{1}{1}+\frac{1}{|\underline{2}}-\frac{1}{|\underline{3}}+\frac{1}{|\underline{4}}-\frac{1}{|\underline{5}}+\frac{1}{|\underline{6}}-\text{etc.}\ .\ .\ .\ -\frac{(-1)^n}{|\underline{n+3}}\right\}.$$

For denominators, we have the same general equation for q_n as for p_n.

Now $$q_1=2,\ q_2=8;$$

$\therefore$, from the last of equations (3),

$$\frac{4.8}{\underline{|5}}-\frac{3.2}{\underline{|4}}=\frac{A}{\underline{|5}};\ \therefore\ A=2;$$

$$\therefore\ \frac{n+2}{\underline{|n+3}}q_n=\frac{3.2}{\underline{|4}}+2\left\{\frac{1}{\underline{|5}}-\frac{1}{\underline{|6}}+\ \dots\ +\frac{(-1)^n}{\underline{|n+3}}\right\}$$

$$=1-2\left\{1-1+\frac{1}{\underline{|2}}-\frac{1}{\underline{|3}}+\frac{1}{\underline{|4}}-\frac{1}{\underline{|5}}+\dots-\frac{(-1)^n}{\underline{|n+3}}\right\}.$$

$$\therefore\ \frac{p_n}{q_n}=\frac{-1+3\left\{1-1+\frac{1}{\underline{|2}}-\ \dots\ -\frac{(-1)^n}{\underline{|n+3}}\right\}}{1-2\left\{1-1+\frac{1}{\underline{|2}}-\ \dots\ -\frac{(-1)^n}{\underline{|n+3}}\right\}}=\frac{-1+3e^{-1}}{1-2e^{-1}}$$

$$=\frac{3-e}{e-2},\ \text{when } n \text{ is indefinitely increased.}$$

EXAMPLES.—LXXII.

1. Prove that $\dfrac{2}{1+}\,\dfrac{1}{\frac{2}{2}+}\,\dfrac{1}{\frac{2}{3}+}\ \dots\ \dfrac{1}{\frac{2}{n-1}+}\,\dfrac{1}{\frac{2}{n}}=\dfrac{2S'}{1-S'}$,

where $S'=\dfrac{1}{1.2}-\dfrac{1}{2.3}+\ \dots\ +(-1)^{n+1}\dfrac{1}{n(n+1)}$.

Hence find the value of the continued fraction when n is infinite.

2. If $\dfrac{p_n}{q_n}$ be the nth convergent to the fraction, $\dfrac{1}{1+}\,\dfrac{2}{2+}\,\dfrac{3}{3+}$ etc., prove that $p_n+q_n=\underline{|n+1}$.

3. Show that $\frac{1}{1+}\,\frac{2}{2+}\,\frac{3}{3+}$ etc., *ad inf.* $=\frac{1}{e-1}$, and $\frac{2}{2-}\,\frac{3}{3-}\,\frac{4}{4-}$ etc. $\frac{n+1}{n+1}=1+1+\underline{|2}+\underline{|3}+\ \ldots\ +\underline{|n}$.

4. Prove that $\frac{a}{1+}\,\frac{2}{2+}\,\frac{16}{5+}\,\frac{54}{10+}$ etc. $\frac{2n^3}{n^2+1}$

$$=a\left\{\frac{1}{2}+\frac{2^2}{3\,\underline{|3}}+\ \ldots\ +\frac{(-2)^n}{(n+1)\,\underline{|n+1}}\right\}.$$

5. If $\frac{p_n}{q_n}$ be the nth convergent to the continued fraction,

$$\cfrac{1}{1^2+\cfrac{1^2+1}{2^2+\cfrac{2^2+1}{3^2+\cfrac{3^2+1}{4^2+\text{ etc.}}}}}$$

show that $p_n-(n^2+1)p_{n-1}=(-1)^{n+1}$.

6. Prove that $\frac{3}{3-}\,\frac{4}{4-}\,\frac{5}{5-}$ etc., *ad inf.* $=2$.

7. Sum the series

$$\frac{1}{1.3}+\frac{2}{1.3.5}+\frac{3}{1.3.5.7}+\ \ldots\ +\frac{n}{1.3.5\ \ldots\ (2n+1)}.$$

8. Prove that

$$\frac{1}{1+}\,\frac{1}{r+}\,\frac{r+1}{r+1+}\,\frac{r+2}{r+2+}\text{ etc.}=1-\frac{1}{r+1}+\frac{1}{(r+1)(r+2)}-\text{ etc.}$$

9. Prove that

$$e^{\frac{1}{r}}=1+\frac{1}{r-}\,\frac{r}{2r+1-}\,\frac{2r}{3r+1-}\ \cdots\ \frac{nr}{(n+1)r+1-}\ \cdots$$

10. Let there be a series of pairs of quantities, a_1, b_1; a_2, b_2; a_3, b_3; . . . a_n, b_n; any pair being formed from the preceding pair in the following manner:—

$$a_n=b_{n-1},\ b_n=a_{n-1}+b_{n-1},$$

prove that when n is infinite $\frac{a_n}{b_n}=\frac{\sqrt{5}-1}{2}$.

11. There are two vases of equal capacity. One, A, is full of water, and the other, B, is half-full of wine. B is filled up by pouring into it part of the contents of A, and then A is replenished by pouring back into it part of the contents of B. After this double operation has been performed n times, prove that the proportion of the quantity of water in the vase A to the original quantity is $\frac{2}{3}-\frac{2}{3}\left(\frac{1}{4}\right)^n$.

12. Two numbers, a and b, being given, two others, a_1, b_1, are formed from them by the formula $a_1=\frac{2a+b}{3}$, $b_1=\frac{a+2b}{3}$, and two more, a_2, b_2, are formed from these in the same way, and so on continually. Find the limits to which a_n and b_n continually tend as n increases without limit, and prove that they are equal.

MISCELLANEOUS EXAMPLES.

1. If $(5\sqrt{2}+7)^n$ be expanded by the Binomial Theorem in powers of $\sqrt{2}$, prove that the square of the sum of the irrational terms differs from the square of some integer by unity.

2. A series $a_1,\ b_1,\ a_2,\ b_2,\ \ldots$ is formed according to the following law:—a_n is an arithmetic mean between a_1 and b_{n-1}, and b_n is an harmonic mean between b_1 and a_{n-1}. Show that $a_n b_n = a_1 b_1$.

3. Prove that $3^{2n+2}-8n-9$ is divisible by 64 for all positive integral values of n.

4. Solve the equations,

$$(1)\ x^4=4(x-1)(1-x-x^2);$$

$$(2)\ 2x^4-4x+1=0;$$

$$(3)\ (x-2y)^2=y-\frac{x}{3}$$

$$3y(4x-15)=x(3x+1).$$

5. Find the condition that the equations,

$$lx^2+my^2+nz^2=0$$

$$ax+by+cz\ \ =0,$$

may lead to only one set values for the ratios $x:y:z$; and show that, if this condition hold, $\frac{lx}{a}=\frac{my}{b}=\frac{nz}{c}$.

6. Prove that

$$\begin{vmatrix} x & x^2 & x^3 & \ldots & x^n \\ x^2 & x^3 & \ldots & \ldots & x \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ x^n & x & x^2 & \ldots & x^{n-1} \end{vmatrix} = -x^n(x^n-1)^{n-1}.$$

7. Sum to n terms the following series,

$$(1)\ \frac{1}{1.5.9}+\frac{1}{5.9.13}+\frac{1}{9.13.17}+\text{ etc.}$$

$$(2)\ 3+31+235+1575+\text{ etc.},$$

(2) being recurring.

8. Show that a recurring series, whose scale of relation is $1-px-qx^2$, will be convergent or divergent, according as x is less or greater than the numerically least root of the equation $1-px-qx^2=0$.

9. If S_r is equal to the sum of the products, r together, of $1, x, x^2, \ldots x^{n-1}$, prove that

$$S_r = S_{n-r} \cdot x^{-\frac{(n-1)(n-2r)}{2}}.$$

10. If $a_1, a_2, a_3, \ldots a_n$ be in H.P., prove that

$$a_1a_2+a_2a_3+\ldots+a_{n-1}a_n=(n-1)a_1a_n,$$

$$\text{and } a_1a_3+a_2a_4+\ldots+a_{n-2}a_n=\frac{(n-2)}{2}(a_1a_{n-1}+a_2a_n).$$

11. Prove that $\left.\begin{array}{l}(x-\omega y)^n=X-\omega Y\\(x-\omega^2y)^n=X-\omega^2Y\end{array}\right\}$ where $\omega^3=1$,

X and Y being rational functions of x and y.

Thence prove that

$(x^2+xy+y^2)^n$ can be put into the form X^2+XY+Y^2, n being integral.

If $n=2,\quad X=x^2-y^2,\qquad Y=2xy+y^2$;

$n=3,\quad X=x^3-3xy^2-y^3,\quad Y=3xy(x+y)$.

12. The equations,
$$\begin{aligned} x^3+y^3+z^3-3xyz&=a^3\\ yz+zx+xy&=b^2\\ x+y+z&=c,\end{aligned}$$

cannot be simultaneously true, unless $c^3-a^3=3cb^2$; and if this holds, they are true for an infinite number of values of x, y, z.

13. Prove that the first $1+m^2$ terms of the expansion for $(1-x)^{1+\frac{1}{m}}$, according to ascending powers of x, can be made greater than any assignable quantity by taking m large enough, if $x-1$ be positive and not a function of m.

14. Prove that the equation,

$$1-\frac{(x-1)(x-2)}{2.3}+\frac{(x-1)(x-2)(x-3)(x-4)}{2.3.4.5}-\ldots$$
$$-\frac{(x-1)(x-2)\ldots(x-4m+2)}{2.3\ldots 4m-1}=0,$$

is satisfied by $x=-1$, and by $x=4n$, where n may have any integral value from 1 up to m inclusive.

15. If $\overline{f(1)}|^2=a$, $\overline{f(2)}|^2=a+f(1)$, ... $\overline{f(n)}|^2=a+f(n-1)$, where $f(n)$ is positive, whatever integer n be, prove that

$$\frac{1+\sqrt{1+4a}-2f(n+1)}{1+\sqrt{1+4a}-2f(1)}<\left(\frac{2}{1+\sqrt{1+4a}}\right)^n,$$

and is positive.

16. If the square of the sum of n real quantities be equal to $\frac{2n}{n-1}$ times the sum of their products, taken two and two together, the n quantities are all equal to one another.

17. Solve the equations,

(α) $x^3-3x+2=0$, (β) $\sqrt{x^2+4x+3}+\sqrt{x^2+2x-3}=x+3$.

18. Prove that the coefficient of x^{2n} in the expansion of

$$\frac{1}{(1-x)(1+x)^4} \text{ is } \frac{(n+1)(4n^2+11n+6)}{6}.$$

19. Given $yz+zx+xy=1$, show that

$$\frac{x}{1-x^2}+\frac{y}{1-y^2}+\frac{z}{1-z^2}=\frac{4xyz}{(1-x^2)(1-y^2)(1-z^2)}.$$

20. Given $y^2-x^2=\alpha y-\beta x$, $4xy=\alpha x+\beta y$, and $x^2+y^2=1$, eliminate x and y, and show that $(\alpha+\beta)^{\frac{2}{3}}+(\alpha-\beta)^{\frac{2}{3}}=2$.

21. Show that

$$(1+x+x^2)(1+x^3+x^6)\ldots(1+x^{3^{n-1}}+x^{2\cdot 3^{n-1}})$$
$$=1+x+x^2+\ldots+x^{3^n-1}.$$

22. Prove that $\begin{vmatrix} 9 & 13 & 17 & 4 \\ 18 & 28 & 33 & 8 \\ 30 & 40 & 54 & 13 \\ 24 & 37 & 46 & 11 \end{vmatrix}=-105.$

23. If $(by-cx)^2=(b^2-ac)(y^2-cz)$, prove that $(bx-ay)^2 =(b^2-ca)(x^2-az)$.

Also if $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-3\right)=\frac{2x}{a}$, show that

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-2\right)\left\{\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-2\right)^2-3\right\}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)=2\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}\right).$$

24. Show that the sum of all the products of the first n natural numbers, three together, is

$$\frac{(n-2)(n-1)n^2(n+1)^2}{48}.$$

25. Investigate whether the following series be convergent or divergent :—

$$(1)\ 1+\frac{1}{2^2}+\frac{2^2}{3^3}+\frac{3^3}{4^4}+\ \ldots\ ad\ inf.$$

$$(2)\ 1+\frac{1}{\sqrt[2]{2}}+\frac{1}{\sqrt[3]{3}}+\frac{1}{\sqrt[4]{4}}+\ \ldots\ ad\ inf.$$

26. Solve the equations,

$$\text{(i.)}\ x^4+\tfrac{1}{4}=x\sqrt{2}\sqrt{x^4-\tfrac{1}{4}};$$

$$\text{(ii.)}\ \begin{cases}(x^2+a^2)(y^2+b^2)=m(xy+ab)^2\\(x^2-a^2)(y^2-b^2)=n(bx-ay)^2.\end{cases}$$

27. If in the scale 12 a square number ends with a single cypher, the preceding digit is 3, and the cube of the square root ends with 60.

28. Find the sums of every fourth terms of the series, $1+x+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+$ etc. *ad inf.*, commencing respectively at the first, second, third, and fourth terms.

29. Show that the series,

$$\frac{2^{2p}}{(1.2)^p}x^2+\frac{3^{3p}}{(1.2.3)^p}x^3+\ \ldots\ +\frac{n^{np}}{\{\underline{|n}\}^p}x^n+\text{ etc.},$$

is convergent, or divergent, according as x is $<$, or $>$, e^{-p}, p being positive.

30. Prove that, if the denominator of a continued fraction be a

prime number, the error made by taking the second last convergent cannot be an exact multiple of the error made by taking the last convergent.

31. Solve the equations,

$$(\alpha)\left.\begin{array}{l}\dfrac{3}{x}+\dfrac{4}{y}+\dfrac{1}{z}=4\\ yz+zx+xy=\dfrac{11}{6}xyz\\ 2xz+3yz=2xy.\end{array}\right\}$$

$$(\beta)\left.\begin{array}{l}\sqrt{\dfrac{x}{y}}+\sqrt{\dfrac{y}{x}}=\dfrac{5}{2}\\ \sqrt{\dfrac{x^2}{y}}+\sqrt{\dfrac{y^2}{x}}=\dfrac{9\sqrt{2}}{2}\end{array}\right\}.$$

32. Prove the following equalities :—

(1) $a^2(b+c)^2+b^2(c+a)^2+c^2(a+b)^2+2abc(a+b+c)$
$$=2(bc+ca+ab)^2.$$

(2) $a^2b^2c^2-a^2(s-a)^4-b^2(s-b)^4-c^2(s-c)^4$
$$+2(s-a)^2(s-b)^2(s-c)^2$$
$$=2\{(s-b)(s-c)+(s-c)(s-a)+(s-a)(s-b)\}^2;$$

where $s=\dfrac{a+b+c}{2}$.

33. If x be a positive integer, prove that $\dfrac{1-2x^x+x^{x+1}}{(1-x)^2}$ is a positive integer.

34. Solve the equations, (1) $x^2+px+2c\sqrt{x^2+px+q^2}=2cq$.

$$(2)\left\{\begin{array}{l}ax^2+bxy+cy^2=5a+2b,\\ cx^2+bxy+ay^2=5c+2b.\end{array}\right.$$

35. Prove that the relation $\phi(x+2)=\phi(x+1)+\phi(x)$ is satisfied, if $\phi(x)$ is a function of the form

$$A\left(\frac{\sqrt{5}+1}{2}\right)^x+B\left(\frac{-\sqrt{5}+1}{2}\right)^x.$$

Using this result, prove that, if a and b are the first and second

terms of a series, such that any term is the product of the two preceding terms, the nth term is $a^r b^s$, where

$$r=\frac{1}{\sqrt{5}}\left\{\left(\frac{\sqrt{5}+1}{2}\right)^{n-2}-\left(\frac{-\sqrt{5}+1}{2}\right)^{n-2}\right\},$$

and s is a similar quantity with $n-1$ as exponent instead of $n-2$.

36. Solve the equation,

$$\sqrt{x^2+2x-1}+\sqrt{x^2+x+1}=\sqrt{2}+\sqrt{3};$$

and the equations,

$$m_1x+n_1y=\frac{m_1}{x}+\frac{n_1}{y}+m,$$

$$m_2x+n_2y=\frac{m_2}{x}+\frac{n_2}{y}+n.$$

37. Show that, if $n-1$, $n+1$ be both prime numbers greater than 5, $n^2(n^2+16)$ will be divisible by 720, and that n will be one of the forms $30t$, $30t\pm12$.

38. Find the values of x, y and z which simultaneously satisfy the following equations :—

$$x+y+z=a,\ y^2+2xy=x^2=z^2+2zx+2yz.$$

39. If $mx_1^2+ny_1^2=a^2$, $mx_2^2+ny_2^2=a^2$ and $mx_1x_2+ny_1y_2=0$, then $x_1^2+x_2^2=\frac{a^2}{m}$, and $y_1^2+y_2^2=\frac{a^2}{n}$.

40. If $\sqrt{x^2+ax-1}+\sqrt{x^2+bx-1}=\sqrt{a}+\sqrt{b}$,

$$\text{then } x=1, \text{ or } \frac{(\sqrt{a}+\sqrt{b})^2+4}{(\sqrt{a}-\sqrt{b})^2-4}.$$

41. Solve the equations

$$yz+zx+xy=3,$$
$$yz(y+z)+zx(z+x)+xy(x+y)=3,$$
$$yz(y^2+z^2)+zx(z^2+x^2)+xy(x^2+y^2)=3.$$

42. Elimate x and y from the equations

$$ax+by=x+y+xy=x^2+y^2-1=0,$$

and show that $\frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{(a-b)^2}$.

43. Prove that

$$\left(\frac{27a^4-18a^2b^2-b^4}{8ab^2}\right)^2+\left\{\frac{(9a^2-b^2)^{\frac{3}{2}}\sqrt{b^2-a^2}}{8ab^2}\right\}^2=b^2.$$

Solve the equation $\dfrac{x}{2}+\dfrac{\overline{x-1}|^{\frac{3}{2}}}{\sqrt{4x-1}}=\dfrac{11}{16}.$

44. Solve the equations, $\left.\begin{aligned} x+y+z&=3a \\ yz+zx+xy&=3a^2 \\ xyz&=a^3 \end{aligned}\right\}.$

45. Solve the equations,

$$x^3+\frac{a^4}{x}=2a^2x;$$

and
$$x^4+y^4-x^2-\sqrt{-1}y^2=0,$$
$$x^2+y^2-x-\sqrt{-1}y=0.$$

46. Solve the equations, $x^3+y^3+z^3=3xyz$, $x-a=y-b=z-c$.

47. Solve the equations

$$6(x^2+y^2+z^2)=13(x+y+z)=\frac{481}{6}, \quad xy=z^2.$$

48. Find the sum of the series, continued to n terms,

$$1.3^2+2.4^2+3.5^2+ \text{ etc.}$$

49. If ${}_nP_r$ denote the number of permutations of n things, taken r together, show that the limit of the expression

$$\log\{{}_nP_1+{}_nP_2+ \;.\;.\;.\; +{}_nP_n\}-\log \underline{|n},$$

when n is indefinitely increased, is unity.

50. Eliminate l, m, n from the equations

$$a^2l^3+b^2m^3+c^2n^3=a'^2l+b'^2m+c'^2n,$$
$$al=bm=cn,$$
$$l^2+m^2+n^2=1.$$

51. If ${}_nP_r$ nenote the number of permutations of n things, taken r together, and $\Sigma({}_nP)$ denote ${}_nP_1+{}_nP_2+ \;.\;.\;.\; +{}_nP_n$, show that

$$\Sigma({}_{n+1}P)=(n+1)\{\Sigma({}_nP)+1\}.$$

52. There are two numbers, a and b. It is required to find n intermediate numbers, a_1, a_2, . . . a_n, so that a_1-a, a_2-a_1, a_3-a_2, . . . a_n-b may form an A. P. with the common difference d.

Determine a_1, a_2, . . . ; and find the limits between which d must lie.

53. What is the limiting value of $x+\frac{1}{x+}\,\frac{1}{x+}\,\ldots$, when x approaches zero.

54. Solve the equations,

$$\left.\begin{aligned} x^2+xy+2y^2&=74 \\ 2x^2+2xy+y^2&=73 \end{aligned}\right\}.$$

55. The coefficient of x^r in the expansion of

$$(1+x)(1+cx)(1+c^2x)\ \ldots\ ,$$

the number of factors being unlimited, and c less than unity, is equal to

$$\frac{c^{\frac{1}{2}r(r-1)}}{(1-c^1)(1-c^2)(1-c^3)\ \ldots\ (1-c^r)}.$$

56. If A_0, A_1, A_2, . . . be the successive coefficients of a binomial raised to an integral power n, show that

$$(A_0-A_2+A_4-\text{ etc.})^2+(A_1-A_3+A_5-\text{ etc.})^2$$
$$=A_0+A_1+A_2+\ \ldots\ +A_n.$$

57. If a, b, c be positive integers, and $a^{\frac{2}{b}}$, $b^{\frac{1}{ac}}$, $c^{\frac{2}{b}}$ be in G. P., show that $a^{\frac{2}{b^n}}$, $b^{\frac{1}{a^nc^n}}$, $c^{\frac{2}{b^n}}$ are also in G. P.

58. A gentleman and his family drink year by year a quantity of sherry, which varies, directly as his income, directly as the square of the mean annual temperature, and inversely as the price of the wine. One year when his income was £600 and the mean temperature 49°, they drank 6 octaves at £8 the octave. Another year, when the mean temperature was 50°, they drank 9 octaves at £10 the octave. What was the gentleman's income in the latter year?

59. If ${}_pC_n$ denote the number of combinations of p things, n

together, where p is a prime number, prove that ${}_{p-1}C_n+(-1)^{n-1}$ is divisible by p.

60. Show that the number of ways, in which mn things can be divided among m persons, so that each shall have n of them, is $\dfrac{\lfloor mn}{(\lfloor n)^m}$.

61. If the odd convergents to a continued fraction be $\dfrac{p_1}{1}, \dfrac{p_3}{q_3}, \ldots \dfrac{p_{2n+1}}{q_{2n+1}}$, the corresponding quotients being $p_1, m_3, m_5 \ldots m_{2n+1}$, prove that the continued fraction is equal to $p_1+\dfrac{m_3}{q_3}+\dfrac{m_5}{q_3q_5}+\ldots+\dfrac{m_{2n+1}}{q_{2n-1}q_{2n+1}}+$ etc.

Give the last term of this series.

62. Solve the equations,

(i.) $1+4x-8x^3+2x^4=0$;

(ii.) $\sqrt{(x^2+a^2)(y^2+b^2)}+\sqrt{(x^2+b^2)(y^2+a^2)}=(a+b)^2$,

$$x+y=a+b.$$

63. There are $p+q$ numbers, $\alpha, \beta, \gamma, \ldots$, of which p are even and q odd. Show that the sum of the products, taken 3 and 3 together, of the quantities $(-1)^\alpha, (-1)^\beta, (-1)^\gamma, \ldots$ etc. is

$$\tfrac{1}{6}\{(q-p)^3-3(q^2-p^2)+2(q-p)\}.$$

64. If $x_n=x(x+1)(x+2)\ldots(x+n-1)$, show that

$$(x+y)_n=x_n+nx_{n-1}y_1+\frac{n(n-1)}{1.2}x_{n-2}y_2+\text{ etc. }\ldots+y_n.$$

65. If $f(r)=\dfrac{\lfloor n}{\lfloor r\,\lfloor n-r}+n\dfrac{\lfloor n}{\lfloor r+1\,\lfloor n-r-1}$

$$+\frac{n(n-1)}{1.2}\frac{\lfloor n}{\lfloor r+2\,\lfloor n-r-2}+\text{ etc.},$$

the

$$f(0)+nf(1)\frac{n(n-1)}{1.2}f(2)+\ldots+f(n)=\frac{(2n+1)(2n+2)\ldots 3n}{\lfloor n}.$$

66. If $p_r = \dfrac{n(n-1)\ .\ .\ .\ (n-r+1)}{\underline{|r}}$, prove that

$$p_1 - \frac{p_2}{2} + \frac{p_3}{3} - \text{etc.} - \frac{(-1)^n p_n}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \ .\ .\ .\ + \frac{1}{n}.$$

67. If $\sqrt{x+a+b} + \sqrt{x+c+d} = \sqrt{x+a-c} + \sqrt{x-b+d}$, then $b+c=0$.

If $x^2+y^2+z^2 = xyz+4$, prove that

$(yz-x)^2 + (zx-y)^2 + (xy-z)^2 = (yz-x)(zx-y)(xy-z)+4.$

68. Solve the equations,

(1) $a^x - 8a^{-x} = 2$;

(2) $10^x = 2$;

(3) $\begin{cases} x^y = y^x, \\ x^p = y^q. \end{cases}$

69. Find the value of $\dfrac{x^4-4x^3+6x^2-4x+1}{x^4-2x^3+2x-1}$ when $x=1$, and also of $\dfrac{\log x}{x-1}$ when $x=1$.

70. Solve the equations, $4 \times 10^{\sqrt{x}} = 25 \times 2^{x}$;

and $\left.\begin{matrix} 2^x = 3^y \\ 2^{y+1} = 3^{x-1} \end{matrix}\right\}$.

71. Solve the equation

$$\frac{5}{x^2-7x+10} + \frac{5}{x^2-13x+40} = x^2 - 10x + 19.$$

72. Having given $u_n = nu_{n-1} + (-1)^n$ and $u_1 = 0$, prove that

$\underline{|n} = u_n + nu_{n-1} + \dfrac{n(n-1)}{1.2} u_{n-2} + \ .\ .\ .\ + \dfrac{n(n-1)}{1.2} u_2 + 1$; and that $\dfrac{u_n}{\underline{|n}}$ is ultimately equal to $\dfrac{1}{e}$, when n increases indefinitely.

73. If α, β be the roots of the equation $ax^2+bx+c=0$, prove that the quadratic equation, whose roots are $\dfrac{\alpha^4}{\beta}, \dfrac{\beta^4}{\alpha}$, is $a^4cx^2 + b(b^4 - 5ab^2c + 5a^2c^2)x + ac^4 = 0.$

74. Show that for all integer values of n, the series

$$1-(n-1)+\frac{(n-2)(n-3)}{1.2}-\frac{(n-3)(n-4)(n-5)}{\underline{|3}}+\ldots=\pm 1 \text{ or } 0,$$

stopping at the first term which vanishes.

75. Prove that the condition, that $ax^2+2bxy+cy^2$ should contain $Lx+My$ as a factor, can be expressed thus,

$$\begin{vmatrix} a & b & L \\ b & c & M \\ L & M & 0 \end{vmatrix}=0.$$

76. Find u, v, x, y from the four equations

$$u+v=a, \quad ux+vy=b, \quad ux^2+vy^2=c, \quad ux^3+vy^3=d.$$

77. Given that $x+\sqrt{x}+2=\dfrac{x^2+x-4}{\sqrt{x}}$, find x.

78. If $a+b+c=0$, prove that $(a^2+b^2+c^2)^2=2(a^4+b^4+c^4)$.

79. A person has £15,000 invested at 4 per cent. He spends £500 a year and invests the remainder at the same rate. Determine in how many years his investment will be trebled.

$$\log 2=\cdot 30103, \quad \log 1\cdot 3=\cdot 11394.$$

80. Sum the series,

$$1.4.7+2.5.8x+3.6.9x^2+\ .\ .\ .\ \text{to } n \text{ terms};$$

$$x+2x^2+7x^3+20x^4+61x^5+\ .\ .\ .\ \textit{ad inf.}$$

81. Prove that

$$\begin{vmatrix} \dfrac{b+c}{a}, & \dfrac{a}{b+c}, & \dfrac{a}{b+c} \\ \dfrac{c+a}{b}, & \dfrac{c+a}{b}, & \dfrac{b}{c+a} \\ \dfrac{c}{a+b}, & \dfrac{c}{a+b}, & \dfrac{a+b}{c} \end{vmatrix}=\frac{2(a+b+c)^3}{(b+c)(c+a)(a+b)}.$$

82. Expand $\dfrac{3-2x}{(2-3x+x^2)^2}$ in ascending powers of x, and find the coefficient of x^r.

83. If n be a positive integer not less than 2, show that

$$a^n(b-c)+b^n(c-a)+c^n(a-b)$$

is divisible by $(a-b)(b-c)(c-a)$.

Determine the quotients for the values 3, 4, and 5 of n. State the general form of the quotient.

84. In the series, $a_1+a_2+\ldots+a_n+$ etc., the $(n+1)$th term is derived from the preceding by the formula, $a_{n+1}=\dfrac{n(n+1)}{2n-a_n}$.

Prove that $a_n=n\dfrac{n-1-(n-2)a_1}{n-(n-1)a_1}$. And if $a_{n+1}=\dfrac{pq}{p+q-a_n}$,

$$\text{then } a_n=\frac{p^nq-cq^np}{p^n-cq^n}, \text{ where } c=\frac{a_0-q}{a_0-p}.$$

85. If n persons agree each to name a number not greater than n, what is the chance (1) that no two persons name the same number, and (2) that they all name the same number?

86. Find all the numbers of 3 digits each, which satisfy the condition that the sum of the digits shall be 15, and the number formed by reversing the order of the digits shall exceed the number by 198.

87. If α, β, γ, δ are the roots of the equation,

$$x^4-px^3+qx^2-rx+s=0,$$

express, in terms of the coefficients, the determinant

$$\begin{vmatrix} \alpha & 1 & 1 & 1 \\ 1 & \beta & 1 & 1 \\ 1 & 1 & \gamma & 1 \\ 1 & 1 & 1 & \delta \end{vmatrix}.$$

88. If x^3+px^2+qx+r, $x^3+p'x^2+q'x+r'$ have a common factor of the form $x-\alpha$, show that

$$\alpha=\frac{(p-p')(rq'-r'q)-(r-r')^2}{(q-q')(r-r')-(p-p')(rp'-r'p)},$$

p, p', q, q', r, r' being subject to an equation of condition.

89. The diameter of a crown and half-crown are ·81, and ·666 inches respectively. Find the least number of coins which

can be placed in a row 9 feet long. Find also the smallest sum which such a row may be made to represent.

90. If n be a prime number and N prime to n, prove that $N^{n^{r+1}-n^r}-1$ is a multiple of n^{r+1}.

91. If a, b, and n are positive integers, and b less than $2a-1$, show that the integral part of $(a+\sqrt{a^2-b})^n$ is an odd number.

92. Is the series $\frac{1}{3}\left(1-\frac{1}{\sqrt{2}}\right)+\frac{1}{4}\left(1-\frac{1}{\sqrt{3}}\right)+$ etc., *ad infinitum*, divergent or convergent?

93. Show that the radix of the scale, in which 49 represents a square number, must be of the form $m(m-3)$.

94. Ten persons each write down one of the digits $0, 1, 2, \ldots 9$ at random. Find the probability of all ten digits being written.

95. If the number of years (ϵ), which a person, whose age is a, may expect to live, be approximately represented by the equation $\epsilon=\frac{2}{3}(80-a)$, what would it cost a man, whose age is 32, to purchase an annuity of £100 for life, interest being reckoned at 4 per cent.?

96. If a_n, b_n be the coefficients of x^n in the expansions of $\frac{2-x}{1-4x+x^2}$ and $\frac{1}{1-4x+x^2}$ respectively; then will $a_n^2-3b_n^2=1$.

97. A and B have equal sums of money (S). A gives B one nth of what he has, then B gives A one nth of what he has, then A gives B one nth of what he has, and so on. Show that after A has given B x times and received x times he will have

$$S\left(1-\frac{1}{n}\right)^{2x}+S\frac{2}{n}\left\{\frac{1-\left(1-\frac{1}{n}\right)^{2x}}{1-\left(1-\frac{1}{n}\right)^2}\right\}.$$

98. A and B throw for a certain stake, each with one die, at one throw, A's die is marked 2, 3, 4, 5, 6, 7, and B's die 1, 2, . . . 6, equal throws dividing the stake. Show that A's expectation is $\frac{47}{72}$ of the stake.

99. If x be a prime and r any number, then x^r+4 cannot be a square number.

100. Develope $\dfrac{36+60x+x^2-50x^3-36x^4-10x^5-x^6}{(6+5x-5x^2-5x^3-x^4)^2}$,

in ascending powers of x as far as x^6.

101. A number taken at random, in the scale of 10, is squared. Show that it is an even chance that the digits in the unit's place of the result is an even number, and that it is 4 to 1 that the digit in the ten's place is an even number.

102. If n be a prime number greater than 2, prove that any number in the scale, whose radix is $2n$, ends in the same digit as its nth power.

103. Solve the equations,

$$x^2(y+z)=a^3,\ y^2(z+x)=b^3,\ xyz=c^3.$$

104. If $P=a_0+a_1\alpha+a_2\alpha^2+\ .\ .\ .\ +a_n\alpha^n$,

$$Q=a_0+a_1\beta+a_2\beta^2+\ .\ .\ .\ +a_n\beta^n,$$

show that, when $a_0=a_n$, $a_1=a_{n-1}$, etc., $P:Q=\alpha^{\frac{n}{2}}:\beta^{\frac{n}{2}}$, where α, β are the roots of the quadratic $x^2+px+1=0$.

105. If ax^3+3bx^2+d and $bx^3+3dx+e$ have a common measure, prove that

$$(ae-4bd)^3=27(ad^2+b^2e)^2.$$

106. Show that, if $x^r+py^r+qz^r$ is exactly divisible by $x^2-(ay+bz)x+abyz$, then $\dfrac{p}{a^r}+\dfrac{q}{b^r}+1=0$.

107. If n be a prime number and p any integer, then $(n^2p^2-1)^{n-1}+1$ and $(np+1)^{n-1}+(np-1)^{n-1}$ have the same remainder when divided by n^2.

If n be a prime and m not divisible by n, prove that

$$m^{2n-1}-1-\frac{\underline{|2n-1}}{\underline{|n-1}\,\underline{|n}}(m-1) \text{ is a multiple of } n.$$

108. There are n dice in the shape of regular tetrahedrons, which have each one side marked with 2, two sides with 1, and the remaining side blank. If they are thrown on a table, show

that the chance of the sum of the numbers on the uncovered sides amounting to $3n$ exactly is $\frac{1}{2^n}\,\frac{1.3.5\ldots(2n-1)}{\underline{|n}}$.

109. Find the scale in which the number 16640 in the common scale appears as 40400.

110. From a box containing three £50 notes, three £20 notes, and three £10 notes, three notes are taken at random and put into a bag. A note is drawn out twice from the bag and replaced. Each time it is found to be a £50 note. Find the probable value of the contents of the bag, supposing (1) that the numbers on the notes drawn were not observed, (2) that they were observed and found to be the same, (3) that they were observed and found to be different.

111. Show that the greatest coefficient in the expansion of $(a_1+a_2+\ldots+a_m)^n$ is $\frac{\underline{|n}}{(\underline{|q})^m(q+1)^r}$, where q is the quotient, and r the remainder, when n is divided by m.

112. Show that

$$\frac{1}{p+n}+\frac{n+1}{(p+n)(p+2n)}+\frac{(n+1)(2n+1)}{(p+n)(p+2n)(p+3n)}+\ldots \textit{ ad. inf.}$$
$$=\frac{1}{p-1}, \text{ if } p>1 \text{ and } p+n>0.$$

113. Examine in what cases the following series is convergent or divergent,

$$2x+\frac{3^4}{2^4}x^2+\frac{4^9}{3^9}x^3+\ldots+\left(\frac{n+1}{n}\right)^{n^2}x^n+\ldots$$

114. Sum the series

$$\frac{1}{1(n+1)}+\frac{1}{2(n+2)}+\frac{1}{3(n+3)}+\ldots \textit{ ad infinitum,}$$

and thence show that, if a_n be the coefficient of x^n in the expansion of $\log(1+x)\log\left(1+\frac{1}{x}\right)$,

$$(-1)^n na_n=1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}.$$

115. In the equation, $x^4-2ax^3+bx^2+cx+f=0$, the sum of two roots is a. Find all the roots, and determine the relation which must exist between the coefficients of the given equation.

116. Find an expression for the series

$$1+nx+\frac{(n-1)(n-2)}{1.2}x^2+\frac{(n-2)(n-3)(n-4)}{1.2.3}x^3+\text{ etc.}$$

The sum of all terms of the series, $1.2+2.3+3.4\ldots+(n-1)n$, which on division by 7 have an odd remainder, and of which $(n-1)n$ is the greatest, is $\dfrac{(n+3)(n^2+6n-4)}{21}$.

117. If $2s=a+b+c$, prove that

$$\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 2(s-b)(s-c) & sc & sb \\ 1 & sc & 2(s-c)(s-a) & sa \\ 1 & sb & sa & 2(s-a)(s-b) \end{vmatrix}$$

$=-16s(s-a)(s-b)(s-c)$.

118. A box contains three Bank of England notes, any of which may be a five-pound, ten-pound, or a twenty-pound note; one is drawn, found to be a five-pound note, and then replaced. What is another draw worth?

119. Show that the sum of the series, $2+6+14+30+$ etc., is $2^{n+2}-(2n+4)$.

120. Prove that the determinant

$$\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$$

is divisible by $a+\omega b+\omega^2 c$, when ω is a cube root of unity.

121. If S_n be the sum of the nth powers of a number of terms of an A. P., of which a is the first and l the last term, and δ their common difference, and if

$$(S+\delta)_n=S_n+nS_{n-1}\delta+\frac{n(n-1)}{1.2}S_{n-2}\delta^2+\frac{n(n-1)(n-2)}{\lfloor 3}S_{n-3}\delta^3+\text{ etc.},$$

the coefficients following the law of the binomial theorem, then

$$(S+\delta)_n-S_n=(l+\delta)^n-a^n.$$

122. Prove that $(1+x)^n(1+x^n)>2^{n+1}x^n$, n being positive.

123. Prove that, $r^{4n}+2nr^{2n-1}>1+2nr^{2n+1}$,

$$\text{and } r^{\frac{n-1}{2}}(r^{\frac{n+1}{2}}+n)>nr^{\frac{n+1}{2}}+1, \text{ when } r>1.$$

124. If $a_1, a_2, \ldots a_n$ be any odd numbers, and if S_r be the sum of their products, taken r together, prove that

$$S_1+S_2+S_3+\ \ldots\ +S_{n-1},$$
$$\text{and } S_1-S_2+S_3+\ \ldots\ +(-1)^nS_{n-1}$$

are both even numbers.

If x be any odd number, prove that

$$1-x+\frac{x(x-1)}{1.3}-\frac{x(x-1)(x-3)}{1.3.5}+\text{ etc. }=0.$$

ANSWERS.

APPENDIX TO PART I.

I. p. x.

1. $x=-(a+b+c)$, $y=bc+ca+ab$, $z=-abc$.

2. $x=1$, $y=2$, $z=3$.

3. $x=3$, $y=\frac{1}{2}$, $z=\frac{2}{5}$.

4. $x=\dfrac{1}{(a-b)(a-c)}$, $y=-\dfrac{1}{(b-a)(b-c)}$, $z=\dfrac{1}{(c-b)(c-a)}$.

5. $x=-\dfrac{11}{40}$, $y=1\dfrac{17}{40}$, $z=3\dfrac{21}{40}$.

6. $x=\dfrac{b^2+c^2-a^2}{2bc}$.

7. $x=\dfrac{a^3}{(a-c)(a-b)}$.

8. $x=\dfrac{n-m}{mn}$.

II. p. xiii.

1. 74. 2. Remainder$=5$. 3. Remainder$=8$.

III. p. xvii.

1. $x>2$, or $<\dfrac{1}{3}$. 2. $\dfrac{2+\sqrt{61}}{6}$. 3. It is $>-7+5\sqrt{2}$, or $<-7-5\sqrt{2}$; Between $\dfrac{8\pm\sqrt{247}}{6}$. 4. $\dfrac{28}{3}$. 5. All between $\dfrac{2\pm\sqrt{11}}{7}$. 6. All between $9\pm4\sqrt{5}$. All between $\dfrac{6\pm3\sqrt{6}}{4}$. 7. $\dfrac{7}{2}$ and $-\dfrac{3}{2}$.

IV. p. xviii.

1. $\frac{n(n+1)(n+2)(n+3)}{4}$.

2. $\frac{2n}{3}(n+1)(2n+1)$.

3. $n(2n+1)^2$.

4. $2n^2(n+1)^2$.

5. $\frac{n(n+1)(2n+7)}{6}$.

6. $\frac{n(n+1)(n+2)(3n+1)}{12}$.

PART II.

I. p. 10.

1. $a^2-b^2-2ab\sqrt{-1}$; $a^3-3ab^2+b(b^2-3a^2)\sqrt{-1}$.

2. $\frac{a+b\sqrt{-1}}{a^2+b^2}$; $\frac{x^2-y^2-2xy\sqrt{-1}}{x^2+y^2}$; $y-x\sqrt{-1}$.

3. $\frac{2(a^2-b^2)}{a^2+b^2}$.

4. $3\sqrt{2}$.

5. $-\frac{1+\sqrt{-3}}{2}$.

6. $1-4\sqrt{-5}$.

7. $\sqrt{2}\{\sqrt{a+\sqrt{a^2+b^2}}+\sqrt[4]{a^2+b^2}\}^{\frac{1}{2}}$.

8. $\sqrt{-1}$; $1-2\sqrt{-2}$.

10. $\sqrt{6+2\sqrt{13}}$.

II. p. 20.

10. x^6+y^6.

16. x must not be $<\frac{11}{7}$.

19. $x+\frac{3}{4}>4$, if x is positive, and >3 or <1,

<4, . . . negative, or between 1 and 3.

Least positive value is $2\sqrt{3}-3$.

III. p. 24.

1. $\dfrac{2n}{(2n-1)(2n+1)}$. 2. $\dfrac{\underline{|2n}}{2^{2n}\{\underline{|n}\}^2}$. 3. $\dfrac{(2n-1)2^n}{\underline{|n+2}}$.

4. $\dfrac{x^{n-1}}{n(n+1)(n+2)}$. 5. $\dfrac{1}{(nx+1)(nx+2x+1)}$.

6. $-(3n-4)x^{n-1}$. 7. 2^n-1. 8. $(3n-2)x^{n-1}$.

9. $(-1)^{n-1}n(a+n-1)x^{n-1}$. 10. $\dfrac{1}{8(n+2)(n+3)}$.

11. $(-1)^{n-1}2.\dfrac{2n+1}{3^n+7}$. 11. $(-1)^{n-1}2\dfrac{2n+1}{3^n+7}$.

12. $\left(\dfrac{(n+1)^2-1}{(n+1)^2+1}\right)x^{n-1}$. 13. $\dfrac{\underline{|n+p-1}}{\underline{|n-1}\,\underline{|p}}$.

14. $(-1)^{n-1}.\dfrac{\underline{|r-n+1}}{\underline{|n}\,\underline{|r-2n+2}}.3^{r-n}$. 15. $\dfrac{1}{(2n+1)\,\underline{|2n-1}}$.

16. $\dfrac{2^{n-1}}{\underline{|n+1}}$.

IV. p. 34.

2. Div. 3. Div. 4. Div. 5. Div. 6. Div. 7. Conv. 8. Div.

V. p. 40.

3. If $x=$ or <1, Div.; if $x>1$, Conv.

4. If $x=$ or >1, Div.; if $x<1$, Conv.

5. If $x<1$, Conv.; if $x=$, or >1, Div.

6. If $x=$, or >1, Div.; if $x<1$, Conv.

7. If $x=$, or >1, Div.; if $x<1$, Conv.

8. If n is positive, but not integral, and $x=$, or <1, Conv.; otherwise Div.

VI. p. 42.

7. When $x=$, or <1.

VII. p. 42.

1. (1.) Div.; (2.) Div.

2. If $a<1$, Conv.; if $a=$, or >1, Div.

VIII. p. 46.

2. $2\left(1+\frac{\overline{\log_e n}|^2}{\underline{|2}}+\frac{\overline{\log_e n}|^4}{\underline{|4}}+\text{etc.}\right)$.   4. $2\left(1+\frac{1}{\underline{|3}}+\frac{1}{\underline{|5}}+\text{etc.}\right)$.

6. $\frac{1}{2}(e+e^{-1}-2)$.   9. $2\left(1-\frac{1}{\underline{|2}}+\frac{1}{\underline{|4}}-\frac{1}{\underline{|6}}+\text{etc.}\right)$.

10. $\sqrt{e}-1$.

IX. p. 50.

9. $1+\frac{1-\log_{10}3}{\log_{10}2}$.

X. p. 55.

1. $qc-q^2a-pq(b-pa)-e=0,$
   $qd-pe-q(b-pa) \quad =0.$ 2. $4q=4(m+1)+p^2$.

3. $(B^2-CA)(D^2-AF)=(BD-EA)^2$.

4. $(2x+y-3)(x-11y+1)$.   5. $(2x^2+x+2)^2-(\sqrt{5}x)^2$.

6. (1.) $4cbd^2=8ad^2+c^3$; (2.) $b^3+8a^2d=4abc$; $1, \frac{1}{2}, -1$.

XI. p. 59.

1. $a+(b-a)x-bx^2+ax^3+(b-a)x^4-bx^5+$ etc.

3. $x=y+\frac{y^2}{2}+\frac{y^3}{4}+$ etc.   4. $x-\frac{3x^2}{\underline{|2}}+\frac{4x^3}{\underline{|3}}-\frac{5x^4}{\underline{|4}}+$ etc.

6. $\frac{x}{3}+\frac{x^3}{3^4}+\frac{x^5}{3^6}+$ etc.

XII. p. 63.

1. If $x<1$, and $p=\frac{9x}{1+2x}$; when p is an integer it is the $(p+1)$th and the pth, when p is not an integer, let q denote its integral part, then it is the $(q+1)$th.

2. If $x<1$, it is the 1st or 2d, according as x <, or >, $\frac{15}{2}$.

3. If $x<\frac{1}{3}$, as in 1, where $p=\frac{3x}{2(1-3x)}$.

4. If $x=1$, the 1st; if $x<1$ the 1st or 2nd, according as x < or > $\frac{7}{12}$.

5. If $x<\frac{2}{3}$ and $p=\frac{3x}{2-3x}$, then as in 1.

6. If $x=$, or $<$, $\frac{2}{3}$, the 1st.

XIII. p. 65.

5. $\frac{|\underline{m+n}}{|\underline{m+n-r}\;|\underline{r}}$.

6. If r is odd, $2(n)_{\frac{r-1}{2}}\cdot(n)_{\frac{r+1}{2}}$; if r is even, $\left\{(n)_{\frac{r}{2}}\right\}^2$.

7. $\frac{|\underline{2n}}{\{|\underline{n}\}^2}$.

XIV. p. 68.

3. $\frac{(n+1)(n+2)}{2}$.

XV. p. 70.

1. $2\left\{2^n+\frac{n(n-1)}{1.2}2^{n-2}3+\frac{n(n-1)(n-2)(n-3)}{\lfloor 4}2^{n-4}3^2+\text{etc.}\right\}-1.$

2. $2\left\{3^n+\frac{n(n-1)}{1.2}3^{n-2}.7+\text{etc.}\right\}-1.$

3. $2\left\{3^n+\frac{n(n-1)}{1.2}3^{n-2}.6+\text{etc.}\right\}-1.$

4. $2\left\{7^n+\frac{n(n-1)}{1.2}7^{n-2}.3^2.5\right.$
$\left.+\frac{n(n-1)(n-2)(n-3)}{\lfloor 4}.7^{n-4}.3^4.5^2+\text{etc.}\right\}-1.$

5. $2\left\{3^n+\frac{n(n-1)}{1.2}3^{n-2}.2^3\right.$
$\left.+\frac{n(n-1)(n-2)(n-3)}{\lfloor 4}3^{n-4}.2^6+\text{etc.}\right\}-1.$

6. $2\left\{5^n+\frac{n(n-1)}{1.2}5^{n-2}.3^2.2+\text{etc.}\right\}-1.$

XVI. p. 71.

1. (1), $2\left\{3^{\frac{n}{2}}+\frac{n(n-1)}{1.2}3^{\frac{n-2}{2}}+\text{etc.}\right\}-1;$

(2), $2\left\{n3^{\frac{n-1}{2}}+\text{etc.}\right\}$

2. (1), $2\left\{6^{\frac{n}{2}}+\frac{n(n-1)}{1.2}6^{\frac{n-2}{2}}2^2+\text{etc.}\right\}-1;$

(2), $2\left\{n.6^{\frac{n-1}{2}}2+\frac{n(n-1)(n-2)}{1.2.3}6^{\frac{n-3}{2}}2^3+\text{etc.}\right\}.$

3. (1), $2\left\{10^{\frac{n}{2}}+\frac{n(n-1)}{1.2}10^{\frac{n-2}{2}}3^2+\text{etc.}\right\}-1;$

(2), $2\{n.10^{\frac{n-1}{2}}3+\text{etc.}\}.$

4. (1), $2\left\{12^{\frac{n}{2}}+\frac{n(n-1)}{1.2}12^{\frac{n-2}{2}}3^2+\text{ etc.}\right\}-1$;

(2), $2\{n.12^{\frac{n-1}{2}}3+\text{etc.}\}$.

5. (1), $2^{\frac{n+2}{2}}\left\{3^n+\frac{n(n-1)}{1.2}3^{n-1}2^2+\text{etc.}\right\}-1$;

(2), $2\{n.18^{\frac{n-1}{2}}.4+\text{etc.}\}$.

6. (1), $2.3^n\left\{5^{\frac{n}{2}}+\frac{n(n-1)}{1.2}5^{\frac{n-2}{2}}2^2+\text{ etc.}\right\}-1$;

(2), $2.3^n\{n.5^{\frac{n-1}{2}}2+\text{etc.}\}$.

7. (1), $2\left\{32^{\frac{n}{2}}+\frac{n(n-1)}{1.2}32^{\frac{n-2}{2}}5^2+\text{etc.}\right\}-1$;

(2), $2\{n.32^{\frac{n-1}{2}}5+\text{etc.}\}$.

8. $2\left\{5^{\frac{n}{2}}+\frac{n(n-2)}{1.2}5^{\frac{n-2}{2}}.6+\frac{n(n-2)(n-4)(n-6)}{\underline{|4}}5^{\frac{n-4}{2}}6^2+\text{etc.}\right\}-1.$

9. $$2^{\frac{n+2}{2}}\left\{6^{\frac{n}{2}}+\frac{n(n-2)}{2^2.2}6^{\frac{n-2}{2}}.35 +\frac{n(n-2)(n-4)(n-6)}{2^4.\underline{|4}}6^{\frac{n-4}{2}}35^2+\text{etc.}\right\}-1.$$

10. $2\left\{23^{\frac{n}{2}}+\frac{n(n-2)}{1.2}23^{\frac{n-2}{2}}2^2.33+\text{etc.}\right\}-1.$

XVII. p. 72.

6. $1+10x+55x^2+200x^3+625x^4+1532x^5+655x^6$.

9. $(1-x)^{-\frac{1}{x}}-1$.

12. If $r<m+1$, $n-m+1$; if $r>m+1$ and $<n+2$, $n-r+1$; if $r>n+2$, 0.

15. $\frac{\underline{|r+5}}{\underline{|5}\,\underline{|r}}$.

XVIII. p. 78.

1. 3. 2. -244. 3. -1.

5. $\lfloor 2n \left\{ \frac{1}{(\lfloor n)^2} + \frac{1}{(\lfloor n-1)^2 \lfloor 2} + \frac{1}{(\lfloor n-2)^2 \lfloor 4} + \ldots + \frac{1}{\lfloor 2n} \right\}$;

$-\lfloor 2n \left\{ \frac{1}{\lfloor 2n-1} + \frac{1}{\lfloor 2 \lfloor 2n-3} + \frac{1}{\lfloor 3 \lfloor 2n-5 \lfloor 2} + \text{etc.} + \frac{1}{\lfloor n \lfloor n-1} \right\}$;

$\lfloor 2n \left\{ \frac{1}{\lfloor n-1 \lfloor n+1} + \frac{1}{\lfloor n-2 \lfloor 2 \lfloor n} + \text{etc.} \right.$

$\left. + \frac{1}{\lfloor 2n-4 \lfloor 3} + \frac{1}{\lfloor 2n-2 \lfloor 2} \right\}$.

6. -104. 7. 99. 8. $-\frac{2601}{256}$. 9. $\frac{3}{2^7} + \frac{3^2}{2^3} + 3$.

10. $\frac{2}{3^2} \cdot \frac{b}{a^{\frac{4}{3}}} \left\{ \frac{2}{3^2} \cdot \frac{b^2}{a} - c \right\}$. 11. $12x^5$. 12. 3. 13. 28.

14. 35. 15. $1 - 8x + 16x^2 + 16x^3 - 56x^4 +$ etc.

16. $1 - \frac{5a}{2}x + \frac{5}{2}\left(\frac{3}{4}a^2 + 1\right)x^2 - \frac{5a}{4}\left(\frac{a^2}{4} + 3\right)x^3 +$ etc.

XXI. p. 86.

2. $(27r + 2p^3 - 9pq)^2 = 4(p^2 - 3q)^3$.

3. (1), $\frac{-q}{4}$; (2), $-p^3 + 3pq - 3r$; (3), $\frac{3r - pq}{r}$. 4. $2, 2, 1$.

5. (1), 3; (2), -3. 6. (1), $x^3 - qx^2 + prx - r^2 = 0$;
(2), $x^3 + (p^2 - 2q)x^2 + (q^2 - 2pr)x - r^2 = 0$;

(3), $x^3 + (3q - p^2)x - \left\{ \frac{(2p^3 - 9pq + 27r)^2 + 4(3q - p^2)^3}{27} \right\}^{\frac{1}{2}} = 0$.

7. $a^2c^2x^2 - (b^2 - 2ac)(a^2 + c^2)x + (b^2 - 2ac)^2 = 0$.

11. $x^2 - 4mnx - (m^2 - n^2)^2 = 0$. 15. $\frac{1}{2}(3 \pm \sqrt{153}), r = -180$.

XXII. p. 89.

1. $q=9$, 3rd root $=1$.
2. $\frac{1}{2}(-3-\sqrt{-31})$, $\frac{1}{2}(3\pm\sqrt{-17})$.
3. $\pm\sqrt{2}$, $-1\pm\sqrt{3}$.
4. $-1-\sqrt{-3}$, 2, 4.
5. $16x^4-8x^2+49=0$.
6. $x^3-12x-65=0$.

XXIII. p. 90.

2. -1, $\dfrac{1\pm\sqrt{-3}}{2}$.
4. -2, $1\pm\sqrt{-3}$.

XXIV. p. 95.

1. $a=\frac{1}{2}$, $b=-4$, $c=4\frac{1}{2}$.
2. (1), $-\dfrac{1}{7(x+1)}+\dfrac{8}{7(x-6)}$;

   (2), $\dfrac{2x+1}{2(x^2+1)}-\dfrac{1}{x-1}+\dfrac{1}{2(x-1)^2}$;

   (3), $-\dfrac{1}{2}-\dfrac{12}{2(2x-3)}-\dfrac{17}{2(2x+3)^2}$;

   (4), $\dfrac{18}{5(x+6)}-\dfrac{3}{5(x+1)}$;

   (5), $\dfrac{1}{b^2-a^2}\left(\dfrac{Ax+B}{x^2+a^2}-\dfrac{Ax+B}{x^2+b^2}\right)$;

   (6), $\dfrac{3}{4(x+1)}+\dfrac{31}{2(x+3)}+\dfrac{19}{x+2}$;

   (7), $\dfrac{3}{2(x-1)}-\dfrac{7}{x-2}+\dfrac{13}{2(x-3)}$.
3. (1), $\dfrac{6}{7(2x-3)}-\dfrac{3}{7(x+2)}+\dfrac{2}{(x+2)^2}$;

   (2), $\dfrac{3}{x-3}+\dfrac{16}{(x-3)^2}+\dfrac{8}{x+5}$;

(3), $\frac{1}{x-1}+\frac{2}{x+1}+\frac{2}{x^2-x+1}$;

(4), $\frac{1}{16(x-1)}+\frac{1}{8(x-1)^2}-\frac{1}{16(x+1)}-\frac{1}{4(x+1)^2}+\frac{1}{4(x+1)^3}$;

(5), $\frac{1}{x+1}+\frac{3}{(x-2)^2}$;

(6), $\frac{-8}{5(x^2+7x+5)}+\frac{8}{5(x+2)}-\frac{3}{(x+2)^2}$;

(7), $2x+\frac{3}{x}-\frac{2}{x^2}-\frac{c}{x+2}+\frac{2}{x+1}$.

4. $\frac{-1}{9(x+2)}+\frac{x-2}{9(x^2+5)}+\frac{x+2}{(x^2+5)^2}$.

5. $\frac{1}{3x-2}+\frac{3}{x+1}$; $\left\{\frac{-3^n}{2^{n+1}}+3(-1)^n\right\}x^n$.

6 (1), $-\frac{4}{3}-\frac{29}{4}x-\text{etc.}-\left\{\frac{7}{3}+2n+\frac{13}{6}\left(-\frac{1}{2}\right)^n\right\}x^n-\text{etc.}$;

(2), $2-\frac{1}{2}x+\text{etc.}+\frac{1}{5}\left\{(-1)^n7+3\left(\frac{3}{2}\right)^n\right\}x^n+\text{etc.}$;

(3), $\frac{1}{3}+\frac{1}{9}x+\ldots+\left\{-\frac{1}{2}+\frac{1}{2^{n-1}}-\frac{7}{6}\frac{1}{3^n}\right\}x^n+\text{etc.}$;

(4), $1+3x+\text{etc.}+\{2^{n+1}-1\}x^n+\text{etc.}$

7. $\frac{1}{b-a}\left\{b-a+\left(\frac{b}{a}-\frac{a}{b}\right)x+\ \ldots\ +\left(\frac{b}{a^r}-\frac{a}{b^r}\right)x^r+\text{etc.}\right\}$.

If $a=b$, the series is $1+\frac{2x}{a}+\frac{3x^2}{a^2}+\text{etc.}$

8. $b^n \left\{1+na+\frac{n(n+1)}{1.2}a^2+\text{etc.},\right.$

$\left.+\frac{n(n+1)\ldots(n+m-2)}{\underline{|m-1}}a^{m-1}.\right\}$

9. $\frac{3}{x-3}-\frac{2}{x-2}$.

XXV. p. 102.

1. £42; $4\frac{16}{21}$ per cent. 2. $\frac{\log 13-\log 2}{1+\log 13-3\log 5}$ years.

3. £900. 4. £16000̇. 5. 4p. c. 6. $£P\left\{1+\frac{r}{q}\right\}^{m}$.

7. 5 p. c. 8. $25r=3-\frac{3}{\left(1+\frac{r}{12}\right)^{120}}$.

9. $£\frac{10000}{3}\cdot\left.\frac{100}{103}\right|^{10}$; $m<\frac{103}{100}$. 10. $100.\frac{B-A}{A}$ p. c.

11. $3\frac{37}{71}$; $\frac{6A}{12+r}\cdot\frac{2+r}{r}$, supposing the interest to be payable at same time as the annuity. 12. $£97\frac{23}{41}$.

13. $\frac{n(n+1)(2n+1)}{6}+\frac{n^2(n+1)(n-1)}{12}r$, r being interest for £1 for 1 year.

15. $\frac{1}{r}\left\{1-\frac{1}{\left(1+\frac{r}{n}\right)^{mn}}\right\}$, r is int. for £1 for 1 year.

16. 8s. $2\frac{2}{3}$d. per cent. nearly.

17. $30\,(21)^3(\overline{21}|^5-\overline{20}|^5):(\overline{21}|^9-\overline{20}|^9)$. 18. $100\frac{2p-q}{p^2}$.

XXVI. p. 108.

1. $2+\frac{1}{16+}\,\frac{1}{5}$.

2. $\frac{1}{2+}\,\frac{1}{16+}\,\frac{1}{5}$.

3. $2+\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{6+}\,\frac{1}{1+}\,\frac{1}{4}$.

4. $\frac{1}{4+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{7}$.

5. $\frac{1}{188+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{8+}\,\frac{1}{2}$.

6. $1+\frac{1}{34+}\,\frac{1}{2+}\,\frac{1}{14}$.

7. $1+\frac{1}{1+}\,\frac{1}{2+}$ etc.

8. $3+\frac{1}{6+}\,\frac{1}{6+}$ etc.

9. $2+\frac{1}{1+}\,\frac{1}{4+}$ etc.

10. $2+\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{4+}$ etc.

11. $4+\frac{1}{8+}\,\frac{1}{8+}$ etc.

12. $5+\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{10+}$ etc.

13. $6+\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{12+}$ etc.

14. $6+\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{12+}$ etc.

15. $9+\frac{1}{1+}\,\frac{1}{18+}$ etc.

16. $10+\frac{1}{5+}\,\frac{1}{10+}\,\frac{1}{5+}\,\frac{1}{10+}$ etc.

17. $\frac{1}{4+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{8+}$ etc.

18. $\frac{1}{4+}\,\frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{8+}$ etc.

19. $4+\frac{1}{7+}\,\frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{13+}\,\frac{1}{3+}\,\frac{1}{7+}\,\frac{1}{1+}\,\frac{1}{8}$.

XXVII. p. 110.

1. (1), 1, 3, 1, 5, 3; $1, \frac{4}{3}, \frac{5}{4}, \frac{29}{23}, \frac{92}{73}$.

(2), 9, 2, 1, 2; 9, $\frac{19}{2}$, $\frac{28}{3}$, $\frac{75}{8}$.

(3), 3, 1, 2; $\frac{1}{3}$, $\frac{1}{4}$, $\frac{3}{11}$. (4), 1, 1, 8; 1, 2, $\frac{17}{9}$.

2. (1), 2, 4, etc.; 2, $\frac{9}{2}$, $\frac{38}{9}$, $\frac{161}{38}$, $\frac{682}{161}$, $\frac{2889}{682}$.

(2), 3, 1, 2, 1, 6, 1; 3, 4, $\frac{11}{3}$, $\frac{13}{4}$, $\frac{89}{27}$, $\frac{102}{31}$.

(3), 4, 2, 8, 2, 8, 2; 4, $\frac{9}{2}$, $\frac{76}{17}$, $\frac{161}{36}$, $\frac{1364}{305}$, $\frac{2889}{646}$.

(4), 8, 1, 1, 1, 16, 1; 11, 12, $11\frac{1}{2}$, $11\frac{2}{3}$, $11\frac{33}{50}$, $11\frac{35}{53}$.

3. Three, $\frac{4}{9}$, $\frac{1}{18}$, 0.

4. (1), 2, 31, 1; $\frac{31}{100}$, $\frac{7}{300}$, $\frac{3}{1300}$, $\frac{1}{2900}$, 0.

(3), $\frac{71}{229}$; $\frac{16}{687}$, $\frac{7}{2977}$, $\frac{2}{6641}$, $\frac{1}{22900}$.

5. $\frac{1}{15}$, $\frac{1}{180}$, $\frac{1}{329\times 36}$, $\frac{1}{329\times 694}$.

8. $\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{87}$; $\frac{1}{2}$, $\frac{1}{3}$, $\frac{3}{8}$, $\frac{4}{11}$.

9. 1, 2, $\frac{3}{2}$, $\frac{38}{25}$, $\frac{41}{27}$, $\frac{79}{52}$, $\frac{515}{339}$, $\frac{1109}{730}$, $\frac{2733}{1799}$.

11. $\frac{1}{7}$, $\frac{3}{22}$, $\frac{151}{1107}$, $\frac{1362}{9985}$, $\frac{1513}{11092}$.

XXVIII. p. 119.

1. Errata. For 107 read 106; for 2763 read 2721.

2. $\frac{41}{29}$. 4. An error $<\frac{1}{21960}$. 5. $2+\frac{1}{2+}\,\frac{1}{4+}$ etc.; $\frac{218}{89}$.
6. $\frac{42}{79}$. 8. 9th and 10th; 10th and 11th; 49th and 54th; 108th and 119th. 9. 7th and 6th; 15th and 13th; 187th and 162nd; 202nd and 175th; 389th and 337th, etc.
10. $\frac{1}{4}, \frac{7}{29}; \frac{8}{33}, \frac{39}{161}$, etc. 11. A kilometre$=1000$ metres.
12. $\frac{31}{8}$. 14. 3.31. 15. $\frac{213}{275}$.

XXX. p. 126.

1. $\frac{1}{15}(9+2\sqrt{39})$; $1, \frac{3}{2}, \frac{10}{7}, \frac{43}{30}, \frac{53}{37}, \frac{149}{104}$. 12. $\frac{211}{80}$.

XXXI. p. 128.

3. 1.

XXXII. p. 132.

1. 6. 2. 5. 4. 13. 5. If t be any positive integer, all in the formulae $x=3+13t$, $y=2+9t$. 7. 5. 9. 28 cr., 20 half-cr. 10. 8. 11. $x=50, 43, 36, 29, 22, 15, 8, 1$, $y=0, 2, 4, 6, 8, 10, 12, 14$. 12. 119. 13. 329. 14. 1000. 15. Men, 4, 15, 26, 37, 48. Women, 84, 65, 46, 27, 8. 16. Of the first, 17, 10, 3, second, 4, 9, 14. 17. 140.

XXXIII. p. 135.

1. (1.) $x=2, 15$, $y=11, 5$. (2.) $x=1$, $y=13$. (3.) $x=15, 12, 9, 6, 3, 0$, $y=1, 6, 11, 16, 21, 26$. (4.) $x=18, 11, 4$, $y=2, 8, 14$. (5.) $x=2$, $y=3$. (6.) $x=2$, $z=1$.
2. (1.) $x=12$, $y=18$. (2.) $x=29$, $y=20$. (3.) $x=37$, $y=13$.

(4.) $x=23$, $y=31$. 3. $x=17, 130$, $y=19, 374$. 4. $x=10, 21, 32, 43$, $y=10, 23, 36, 49$. 5. $x=90$, $y=89$. 6 The first; the second; 3, 5, or 7 crowns and receives from him 2, 7 or 12 florins.

XXXIV. p. 136.

1. $y=1$, $x=3, 7, 11$, $z=6, 3, 0$; $y=6$, $x=1, 5, 9$, $z=6, 3, 0$; $y=11$, $x=3, 7$, $z=3, 0$; $y=16$, $x=5, 1$, $z=0, 3$; $y=21$, $x=3$, $z=0$; $y=26$, $x=1$, $z=0$.

2. $z=0$, $x=23+7t$, $y=3-3t$, $t=0, -1, -2, -3$; $z=1$, $x=12+7t$, $y=7-3t$, $t=-1, 0, 1, 2$; $z=2$, $x=8+7t$, $y=6-3t$, $t=-1, 0, 1, 2$; $z=3$, $x=7+7t$, $y=4-3t$, $t=0, -1$; $z=4$, $x=6$, $y=2$; $z=5$, $x=5$, $y=0$.

3. $z=1$, $x=2+2t$, $y=10-t$, $t=-1, 0, \ldots 10$; $z=3$, $x=1+2t$, $y=8-t$, $t=0, 1, \ldots 8$; $z=5$, $x=0+2t$, $y=6-t$, $t=0, 1, \ldots 6$; $z=7$, $x=1+2t$, $y=3-t$, $t=0, 1, 2, 3$; $z=9$, $x=0$, $y=1$.

4. $x=0$, $y=6+t$, $z=0-2t$, $t=0, -1, \ldots -6$; $x=1$, $y=4+t$, $z=1-2t$, $t=0, -1, \ldots -4$; $x=2$, $y=3, 2, 1, 0$, $z=0, 2, 4, 6$; $x=3$, $y=1, 0$, $z=1, 3$; $x=4$, $y=0$, $z=0$.

5. $x=0$, $y=18+t$, $z=2-11t$, $t=0, -1, \ldots -18$; $x=1$, $y=15+t$, $z=4-11t$, $t=0, -1, \ldots -15$; $x=2$, $y=12+t$, $z=6-11t$, $t=0, -1, \ldots -12$; $x=3$, $y=9+t$, $z=8-11t$, $t=0, -1, \ldots -9$; $x=4$, $y=6+t$, $z=10-11t$, $t=0, -1, \ldots -6$; $x=5$, $y=3+t$, $z=12-11t$, $t=0, -1$; $2, -3$; $x=6$, $y=1, 0$, $z=3, 14$.

6. 25. 7. 1. 8. 10. 9. Yes.

XXXV. p. 137.

1. $x=1-3t$, $y=51+7t$, $z=63-13t$, $t=0, -1, -2, \ldots -7$.
2. No positive integral solution.
3. $x=4+t$, $y=3t$, $z=1+5t$, $t=0, 1, 2$, etc.
4. 6. 5. Bullocks, 0 or 7; sheep, 100 or 23; geese, 0 or 70.

XXXVI. p. 138.

1. 24. 2. 299, 398, 389, 497, 488, 479, etc.... 992, 983, 929.
3. 36. 4. 15. 5. $1147+1624t$, $t=0, 1, 2$, etc.
6. $211-595t$, $t=0, -1, -2$, etc. 7. 56, 44.
8. $x=3+2t$, $y=23+15t$, $t=0, 1, 2$, etc. 9. 5.
10. $n=1$, $x=1, 19$, $y=1, 19$; $n=2$, $x=13$, $y=6$; $n=6$, $x=17$, $y=2$. 11. 3. 12. $x=485+19t$, $y=582+23t$, $t=-25, -24, \ldots -1, 0, +1$, etc.
13. $\frac{2}{3}, \frac{3}{4}, \frac{4}{5}$. 14. 13. 15. 30.

XXXVII. p. 144.

1. $x=24$, $y=5$; $x=\dfrac{(24+5\sqrt{23})^n+(24-5\sqrt{23})^n}{2}$,

$y=\dfrac{(24+5\sqrt{23})^n-(24-5\sqrt{23})^n}{2\sqrt{23}}$, n is any positive integer.

2. $x=\dfrac{(33+8\sqrt{17})^n+(33-8\sqrt{17})^n}{2}$,

$y=\dfrac{(33+8\sqrt{17})^n-(33-8\sqrt{17})^n}{2\sqrt{17}}$, n is any positive integer.

3. $x=\dfrac{(4+\sqrt{15})^n+(4-\sqrt{15})^n}{2}$, $y=\dfrac{(4+\sqrt{15})^n-(4-\sqrt{15})^n}{2\sqrt{15}}$,

n is any positive integer.

4. $x=\dfrac{(2+\sqrt{5})^n+(2-\sqrt{5})^n}{2}$, $y=\dfrac{(2+\sqrt{5})^n-(2-\sqrt{5})^n}{2\sqrt{5}}$,

n is any positive even integer.

5. $x=\dfrac{(15+4\sqrt{14})^n+(15-4\sqrt{14})^n}{2}$,

$y=\dfrac{(15+4\sqrt{14})^n-(15-4\sqrt{14})^n}{2\sqrt{14}}$, n is any positive integer.

6. $x=\dfrac{(7+4\sqrt{3})^n+(7-4\sqrt{3})^n}{2}$,

$y=\dfrac{(7+4\sqrt{3})^n-(7-4\sqrt{3})^n}{4\sqrt{3}}$ n is any positive integer.

7. $x=\dfrac{(18+5\sqrt{13})^n+(18-5\sqrt{13})^n}{2}$,

$y=\dfrac{(18+5\sqrt{13})^n-(18-5\sqrt{13})^n}{2\sqrt{13}}$, n is any positive odd integer.

8. $h=\dfrac{(8+3\sqrt{7})^n+(8-3\sqrt{7})^n}{2}$,

$k=\dfrac{(8+3\sqrt{7})^n-(8-3\sqrt{7})^n}{2\sqrt{7}}$, n is any positive integer,

$x=3h\pm 7k$, $y=3k\pm h$, both upper, or both lower, signs being taken.

9. $h=\dfrac{(18+5\sqrt{13})^n+(18-5\sqrt{13})^n}{2}$,

$k=\dfrac{(18+5\sqrt{13})^n-(18-5\sqrt{13})^n}{2\sqrt{13}}$, n is any positive even integer; $x=7h\pm 26k$, $y=7k\pm 26k$, both upper, or both lower, signs being taken.

10. $x=1$, $y=3$. 11. $x=0, 6$, $y=2, 8$. 12. $x=2$, $y=1$.

13. $x=3, 0$, $y=4, 8$. 14. No solution.

15. $x=\dfrac{(4+\sqrt{17})^n+(4-\sqrt{17})^n}{2}$,

$y=\dfrac{(4+\sqrt{17})^n-(4-\sqrt{17})^n}{2\sqrt{17}}$, n is any positive odd integer.

16. $x=\dfrac{(170+13\sqrt{19})^n+(170-13\sqrt{19})^n}{2}$,

$y=\dfrac{(170+13\sqrt{19})^n-(170-13\sqrt{19})^n}{2\sqrt{19}}$, n is any positive integer. $x^2-19y^2=-1$ has no solution.

XXXVIII. p. 151.

1. (1). $1-2x+x^2$; (2). $\dfrac{1}{(1-x)^2}$; (3). $(n+1)x^n$;

(4). $\dfrac{1-(n+1)x^n+nx^{n+1}}{(1-x)^2}$.

XXXVIII. p. 151,—*continued.*

2. (1). $(1+x)^2$; (2). $\dfrac{1-x}{(1+x)^2}$; (3). $(-1)^n(2n+1)x^n$;

(4). $\dfrac{1-x}{(1+x)^n}-(-1)^n\dfrac{(2n+1)x^n+(2n-1)x^{n+1}}{(1+x)^n}$.

3. (1). $1-10x+21x^2$; (2). $\dfrac{1+x}{(1-3x)(1-7x)}$;

(3). $(2.7^{n-1}-3^{n-1})x^{n-1}$;

(4). $\dfrac{1+x+(3^n-2.7^n)x^n+21(2.7^{n-1}-3^{n-1})x^{n+1}}{1-10x+21x^2}$.

4. (1). $1-\dfrac{5x}{6}+\dfrac{x^2}{6}$; (2) $\dfrac{2(30-23x)}{6-5x+x^2}$;

(3). $\left(\dfrac{26}{3^{n-1}}-\dfrac{1}{2^{n-5}}\right)x^{n-1}$; (4). $S_n\left(1-\dfrac{5x}{6}+\dfrac{x^2}{6}\right)$

$=10-\dfrac{23}{3}x+\left(\dfrac{26}{3^n}-\dfrac{1}{2^{n-4}}\right)x^n+\dfrac{1}{6}\left(\dfrac{26}{3^{n-1}}-\dfrac{1}{2^{n-5}}\right)x^{n+1}$.

5. (1). $u_n-2u_{n-1}+3u_{n-2}$; (3). $\dfrac{3\sqrt{-2}+2}{4}(1-\sqrt{-2})^{n-1}$

$+\dfrac{2-3\sqrt{-2}}{4}(1+\sqrt{-2})^{n-1}$;

(4). $\dfrac{3}{2}-\dfrac{3-\sqrt{-2}}{4}(1-\sqrt{-2})^n-\dfrac{3+\sqrt{-2}}{4}(1+\sqrt{-2})^n$.

6. (1). $1+\dfrac{x}{2}-\dfrac{x^2}{2}$; (2). $\dfrac{2-4x}{2+x-x^2}$;

(3). $\left\{2(-1)^{n-1}-\dfrac{1}{2^{n-1}}\right\}x^{n-1}$; (4). $S_n\left(1+\dfrac{x}{2}-\dfrac{x^2}{2}\right)$

$=1-2x-\left\{2(-1)^n-\dfrac{1}{2^n}\right\}x^n+\left\{(-1)^n+\dfrac{1}{2^n}\right\}x^{n+1}$.

7. (1). $u_n+\dfrac{2u_{n-1}}{3}-\dfrac{u_{n-2}}{3}$; (3). $(-1)^n-\dfrac{1}{3^{n-1}}$;

(4). $-\dfrac{1}{2}-(-1)^n+\dfrac{1}{2.3^{n-1}}$.

8. (1). $(1-x)^2$; (2). $\dfrac{1+x}{(1-x)^3}$; (3). n^2x^{n-1};

(4). $S_n(1-x)^3=1+x-(n+1)^2x^n+(2n^2+2n-1)x^{n+1}-n^2x^{n+2}$.

9. (1). $1+3x-x^2-x^3$; (2). $\dfrac{1-10x^2}{1+3x-x^2-x^3}$;

(3). $\left\{-\dfrac{9}{8}+\dfrac{9}{4}(-1)^{n-1}-\dfrac{1}{8}(-3)^{n-1}\right\}x^{n-1}$;

(4). $S_n(1+3x-x^2-x^3)=1-10x^2-\left\{-\dfrac{9}{8}+\dfrac{9}{4}(-1)^n\right.$

$\left.-\dfrac{1}{8}(-3)^n\right\}x^n-\left\{\dfrac{9}{2}+9(-1)^{n+1}-\dfrac{1}{4}(-3)^{n+1}\right\}x^{n+1}$

$-\left\{-\dfrac{27}{8}+\dfrac{27}{4}(-1)^{n-1}-\dfrac{3}{8}(-3)^{n-1}\right\}x^{n+2}$.

10. (1). $1-2x-x^2+2x^3$; (2). $\dfrac{3-x-6x^2}{1-2x-x^2+2x^3}$;

(3). $\left\{\dfrac{4}{3}\cdot 2^{n-1}-\dfrac{1}{3}(-1)^{n-1}+2\right\}x^{n-1}$;

(4). $S_n(1-2x-x^2+2x^3)=3-x-6x^2$

$-\left\{\dfrac{4}{3}2^n-\dfrac{1}{3}(-1)^n+2\right\}x^n-\left\{(-1)^n-2\right\}x^{n+1}$

$+\left\{\dfrac{4}{3}2^n-\dfrac{2}{3}(-1)^n+4\right\}x^{n+2}$.

11. (1). $(1+x)^2$; (2). $\dfrac{2+3x}{(1+x)^2}$; (3). $(-1)^{n-1}(3-n)x^{n-1}$;

(4). $S_n(1+x)^2=2+3x-(-1)^n(2-n)x^n$

$+(-1)^{n-1}(3-n)x^{n-1}$.

12. (1). $u_n-10u_{n-1}+21u_{n-2}=0$; (3). $2.7^{n-1}-3^{n-1}$;

(4). $\dfrac{1}{6}-\dfrac{3^n}{2}+\dfrac{7^n}{3}$.

XXXVIII. p. 151,—*continued.*

13. If n be odd, $\left\{\frac{2^{n+2}}{3}-\frac{3}{5}(-1)^{\frac{n-1}{2}}\right\}x^{n-1}$; if n be even, $\frac{1}{5}\left\{2^{n+2}-(-1)^{\frac{n-2}{2}}\right\}x^{n-1}$.

14. (1). $\{(-4)^{n-1}-3^{n-1}\}x^{n-1}$; $\quad \frac{2-7x}{1-7x+12x^2}$.

(2). $\{4^{n-1}+(-3)^{n-1}\}x^{n-1}$; $\quad \frac{2-x}{1-x-12x^2}$.

16. $u_n-12u_{n-1}+32u_{n-2}=0$; $2^{3r-4}+2^{2r-3}$; $\frac{2^{3n}}{14}+\frac{2^{2n}}{6}-\frac{5}{21}$.

17. $u_n-3u_{n-1}+4u_{n-3}=0$; $2-\frac{4}{3}(-1)^n-\frac{2^{n+1}}{3}+n.2^{n-1}$.

18. $a=-2$, $\beta=1$, $S_n(1-x)^2=1+x-(2n+1)x^n$
$+(2n-1)x^{n+1}$.

19. $x^{\frac{n(n-1)}{2}}+1$.

XXXIX. p. 155.

1. $\frac{n(2n+1)(n+1)}{6}$ 2. If n is even, $-2n(n+2)$; odd, $2n(n+2)-3$.

3. $n^2(2n^2-1)-1$.

4. If n is even, $-8n^3+12n^2-5n$; odd, $24n^2-47n+24$.

5. $\frac{n^5}{5}+\frac{n^4}{2}+\frac{n^3}{3}-\frac{n}{30}+1$. 6. $\frac{n(n+1)}{12}(3n^2+23n+46)$.

7. If n is even, $-n-1$; odd, $\frac{(n+1)(n+2)}{2}$.

8. $\frac{n(n+1)(n+4)(n+5)}{4}$.

9. $na^3+\frac{3a^2b}{2}n(n-1)+\frac{ab^2}{2}n(n-1)(2n-1)+\frac{n^2(n-1)^2}{4}b^3.$

10. $\frac{n(n+1)}{12}(n^2+3n+3).$

11. $na^3+a^2b\frac{n(3n+1)}{2}+ab^2\frac{n(n+1)(2n+1)}{2}+b^3\frac{n(n^2-1)(n+2)}{4}.$

12. $\frac{1}{2}n(n+1)(n+2)(3n+1).$

13. $\frac{n^5}{5}+\frac{3n^4}{4}+\frac{7n^3}{3}+n^2-\frac{n}{30}-3.$

14. $\frac{2}{3}n(n+1)(n+2)(3n+7).$ 15. $\frac{1}{6}n(n+1)(n+2).$

16. (1). $2n^3+\frac{7}{2}n^2+\frac{1}{2}n.$ (2). If n is even,

$\frac{1}{6}n(n+1)(3n^2+n+1)-\frac{1}{3}n(n+2)(12n^2+23n-1),$

odd, $\frac{1}{6}n(n+1)(3n^2+n-1)-\frac{1}{3}(n^2-1)(12n^2-n-4).$

17. $\frac{n^4-2n^3+7n^2-6n+8}{4}$; $\frac{n^5}{20}+\frac{5n^3}{12}+\frac{23n}{15}.$

18. $\frac{n}{6}(2n^2+3n-5).$

XL. p. 159.

1. $\frac{1}{4b}(a+\overline{n-1}b)(a+nb)(a+\overline{n+1}b)(a+\overline{n+2}b)$

$-\frac{a}{4b}(a^2-b^2)(a+2b).$

2. $\frac{2}{3}n(n+1)(n+2).$ 3. $\frac{1}{4}n(n+1)(n+2)(n+3).$

4. $\frac{1}{5}n(n+1)(n+2)(n+3)(n+4).$

XL. p. 159—*continued.*

5. $\frac{1}{8}(4n^2-1)(2n+3)(2n+5)+\frac{15}{8}$.

6. $\frac{1}{12}(3n-1)(3n+2)(3n+5)(3n+8)+\frac{20}{3}$.

8. $\frac{1}{10}n(n+1)(n+2)(n+3)(2n+3)$.

9. $\frac{1}{6}n(n+1)(2n+7)$.

10. $\frac{1}{4}n(n+1)(n+2)(n+3)+\frac{1}{3}n(n+1)(2n+7)$.

11. $\frac{1}{12}(n+4)(n+5)(3n^2+13n-6)-10$.

12. $\frac{1}{12}(3n-2)(3n+1)(3n+4)(3n+7)-\frac{14}{3}$.

13. $\frac{1}{5}n(n^2-1)(n+2)(n+3)$.

XLI. p. 163.

1. $\frac{1}{4}-\frac{1}{2(n+1)(n+2)}$; $\frac{1}{4}$.

2. $\frac{1}{ab}-\frac{1}{b(a+nb)}$; $\frac{1}{ab}$.

3. $\frac{1}{32}-\frac{1}{16(n+1)(n+2)}$.

4. $\frac{1}{12}-\frac{1}{4(n+3)}$.

5. $\frac{1}{24}-\frac{1}{6(3n+1)(3n+4)}$.

6. $\frac{1}{40}-\frac{1}{8(4n+1)(4n+5)}$.

7. $\frac{1}{18}-\frac{1}{3(n+1)(n+2)(n+3)}$.

8. $\frac{1}{2}-\frac{1}{4n+2}$.

9. $\frac{1}{90}-\frac{1}{6(2n+1)(2n+3)(2n+5)}$.

10. $1-\frac{1}{n+1}$.

11. $\frac{3}{4}-\frac{2n+3}{2(n+1)(n+2)}$; $\frac{3}{4}$.

12. $\frac{11}{96}-\frac{2n^2+10n+11}{4(n+1)(n+2)(n+3)(n+4)}$; $\frac{11}{96}$.

13. $\dfrac{5}{36}-\dfrac{3n+5}{6(n+1)(n+2)(n+3)}$; $\dfrac{5}{36}$.

14. $\dfrac{5}{4}-\dfrac{4n+5}{2(n+1)(n+2)}$; $\dfrac{5}{4}$.

15. $\dfrac{1}{4}-\dfrac{2n+3}{2(n+2)(n+3)}$.

16. $\dfrac{1}{8}-\dfrac{4n+3}{8(2n+1)(2n+3)}$.

17. $\dfrac{17}{96}$.

18. $\dfrac{n(n+1)}{4(n+2)}$.

19. $\dfrac{m+1}{(m-1)\,\underline{|m}}$.

20. $\dfrac{1}{(n-2)\,\underline{|n-1}}$.

21. $\dfrac{1}{8}-\dfrac{1}{2(n+1)(n+2)}$.

22. $\dfrac{n(n+1)}{n+2}$.

XLII. p. 166.

1. $2a\dfrac{\overline{2a|}^{n}-1}{2a-1}-\dfrac{a^n-1}{a-1}$; $\dfrac{2a}{1-2a}-\dfrac{1}{1-a}$, if $a<\dfrac{1}{2}$.

2. $\dfrac{1+x^2}{(1-x)^3}-\dfrac{(n^2+n+1)x^n-2n^2x^{n+1}+(n^2-n+1)x^{n+2}}{(1-x)^3}$;

$\dfrac{1+x^2}{(1-x)^3}$, if $x<1$.

3. $2x\dfrac{1-x^n}{(1-x)^3}-n\dfrac{(n+3)x^{n+1}-2(n+2)x^{n+2}+(n+1)x^{n+3}}{(1-x)^3}$;

$\dfrac{2x}{(1-x)^3}$, if $x<1$.

4. $\dfrac{1}{r^{n-1}}\,\dfrac{r^n-1}{(r-1)^2}-\dfrac{n}{r^n(r-1)}$; $\dfrac{r}{(r-1)^2}$, if $r>1$.

5. $\dfrac{e^x}{e^{2x}-1}+\dfrac{2}{e^{(2n-3)x}}\,\dfrac{e^{(2n-2)x}-1}{(e^{2x}-1)^2}-\dfrac{2n-1}{e^{(2n-1)x}(e^{2x}-1)}$;

$\dfrac{e^{3x}+e^x}{(e^{2x}-1)^2}$, if x is positive.

6. $\dfrac{3^n-1}{4.3^{n-1}}-\dfrac{n}{2.3^n}$; $\dfrac{3}{4}$.

7. $\dfrac{256}{3}+\dfrac{512}{9}\,\dfrac{4^{n-1}-1}{4^{n-1}}-\dfrac{256}{3}\,\dfrac{2n-1}{4^n}$; $142\dfrac{2}{9}$.

XLII. p. 166—*continued.*

8. $\frac{144x+162x^2+62x^3}{(1+x)^3}-6x^4\frac{1-(-x)^{n-3}}{(1+x)^4}$

$$+(-1)^{n-1}\frac{(n^3+19n^2+106n+138)x^{n+1}+(2n^3+35n^2+171n+168)x^{n+2}+(n^3+16n^2+71n+56)x^{n+}}{(1+x)^3}.$$

$$\frac{144+162x^2+62x^3}{(1+x)^3}-\frac{6x^4}{(1+x)^4}, \text{ if } x^2<1.$$

9. $4-\frac{n^2+5n+8}{2^{n+1}}$; 4. 10. $\frac{1}{4}-(-1)^n\frac{2n^2+1}{4}$.

11. $(-1)^{n-1}(8n^3+30n^2+52n+9)$.

12. $-\frac{1}{9}+\frac{(-2)^{n+1}}{9}-(-2)^n\frac{2n-1}{3}$.

13. $\frac{28}{27}+\left(-\frac{1}{2}\right)^{n-1}\frac{9n^2+30n+14}{27}$.

14. $\frac{1+4x+8x^2+5x^3}{(1-x)^3}+6x^4\frac{1-x^{n-4}}{(1-x)^4}$

$$-\frac{(n^3+3n^2+3n-6)x^n-(2n^3+3n^2-3n-1)x^{n+1}+(n^3-1)x^{n+2}}{(1-x)^3}.$$

15. $\frac{1+x}{(1-x)^2}-\left(\frac{3x-1}{1-x}+2n\right)\frac{x^{n-1}}{1-x}$.

16. $4-\overline{\tfrac{1}{2}}\Big|^{n-2}-(n-1)\overline{\tfrac{1}{2}}\Big|^{n-1}$

17. $\frac{2}{9}+\frac{1}{9}\left(\frac{1}{2}\right)^{n-3}-\frac{2n-1}{3}\left(\frac{1}{2}\right)^{n-1}$.

18. $\frac{2-2x^n}{(1-x)^3}-\frac{n(n+3)x^n}{(1-x)^2}+\frac{n(n+1)x^{n+1}}{(1-x)^2}$.

19. $1+\frac{r}{1-x}\cdot\frac{1-r^{n-1}}{1-r}+\frac{x^2r(1-r^{n-1})}{(1-x)^2(1-r)}-\frac{x^2r(1-x^{n-1}r^{n-1})}{(1-x)^2(1-xr)}$

$$-\frac{x}{(1-x)(1-xr)}\left\{xr+\frac{x^2r^2}{1-xr}\left(1-\overline{xr}|^{n-2}\right)-(n-1)\overline{xr}|^{n}\right\}.$$

XLIII. p. 171.

2. $\dfrac{n}{(a+2b+3c)\,(\overline{n+1}\,a+\overline{n+2}\,b+\overline{n+3}\,c)}$.

3. $\dfrac{1}{120}n\,(n+1)\,(n+2)\,(4n+1)$.

4. $\dfrac{1}{6}n\,(n+1)\,(n+2)-\dfrac{1}{6}\,(n-r)\,(n-r+1)\,(n-r+2)$.

5. 185. 6. 23579. 7. 8610. 8. $\sqrt{e}-1$.

10. $\dfrac{5}{6}-\dfrac{3n+5}{(n+2)\,(n+3)}$. 11. $\left(\dfrac{2}{3}\right)^{\frac{3}{2}}$.

12. $\left(\dfrac{1}{x}+\dfrac{1}{x^2}\right)\log_e(1+x)-\dfrac{1}{x}$. 14. $\dfrac{1}{(n-2)\,\lfloor n-1}$.

15. $\dfrac{n^2+3n-2x^n}{1-x}+\dfrac{2nx^{n+1}}{(1-x)^2}+\dfrac{2x^n-2x}{(1-x)^3}$.

XLIV. p. 174.

1. $3^2.5^2$. 2. $3.11.31$. 3. 17^2. 4. $3^2.5.17.13$. 5. $2^3.3^2.7$.
6. $3^2.13.19^2$. 7. $2^2.3^2.73$. 8. $11.23.29.37$. 9. $2^3.5.7^2.3$.
10. $2^2.3^2.7^2$. 11. $3^2.5.37$. 12. $2.3^4.5.7$. 13. $7^2.89$.
14. $2^4.5^2.11^2$. 15. $2^6.3^2.5$. 16. $2^3.5^2.9^2$. 17. $3^4.5^2.7^2$.

XLV. p. 178.

1. 15. 2. 18. 3. 2205. 4. 147.

XLVI. p. 180.

1. (1), 1, 3, 3^2; 5, 5.3, 5.3^2; 5^2, $5^2.3$, $5^2.3^2$. (2), 9. (3), 403. (4), 5. (5), 2.

XLVI. p. 180—*continued.*

2. (1), 1, 3, 3^2; 13, 13.3, 13.3^2; 19, 19.3, 19.3^2; 19^2, $19^2.3$, $19^2.3^2$; 19.13, 19.13.3, $19.13.3^2$; $19^2.13$, $19^2.13.3$, $19^2.13.3^2$. (2), 18. (3), 74676. (4), 9. (5), 4.

3. (1), 1, 2, 2^2; 3, 3.2, 3.2^2; 3^2, etc.; 7, etc.; 7.3, etc.; 7.3^2, etc.; 7^2, etc.; etc. (2), 27. (3), 1596. (4), 14. (5), 4.

4. 1, 2, 2^2, 2^3, 2^4; 11, 11.2, 11.2^2, 11.2^3, 11.2^4; 11^2, etc.; 5, etc.; etc. (2), 45. (3), 127813. (4), 23. (5), 4.

XLVII. p. 182.

18. $7n+4$. 21. $(7n\pm2)^2$.

XLVIII. p. 186.

1. 96. 2. 600. 3. 504. 4. 400.

LII. p. 193.

5. 5^7; 25^3. 6. 3^6; 6^6.

LIII. p. 200.

1. $\frac{1}{6}$; $\frac{1}{3}$. 2. $\frac{1}{13}$. 4. 7 to 11. 5. $\frac{3}{52}$. 6. $\frac{3}{25}$. 7. $\frac{17}{60}$.

8. 6 white, 4 black, 2 red. 9. $\frac{2}{5}$; $\frac{3}{5}$. 10. $\frac{5}{12}$.

11. 5 to 1; For.

LIV. p. 203.

1. $\frac{25}{121}$; $\frac{30}{121}$; $\frac{60}{121}$. 2. 1 : 2. 3. $\frac{5}{9}$. 4. $\frac{5}{54}$. 5. $\frac{1}{6}$.

6. $\frac{2}{11}$; $\frac{6}{11}$. 7. $\frac{91}{216}$; $\frac{25}{72}$; $\frac{2}{27}$; $\frac{5}{72}$. 8. $\frac{5}{36}$; $\frac{1}{12}$. 9. $\frac{3}{250}$.

10. $\frac{1}{8}$; $\frac{1}{2}$. 11. $\frac{91}{216}$. 13. £1. 2*s*. 14. $\frac{2}{5}$.

15. $\frac{1}{4}$. 16. $\frac{2}{11}$; $\frac{5}{33}$; $\frac{15}{22}$. 17. $\frac{5}{9}$.

LV. p. 205.

1. $\frac{2}{7}$; 6 to 1 against. 2. $\frac{2}{11}$. 3. 13 to 1 against.

4. $\frac{1}{15}$; $\frac{1}{5}$. 5. $\frac{\underline{|20}\,\underline{|39}}{\underline{|7}\,\underline{|52}}$; $\frac{57}{3185}$; $\frac{\underline{|20}}{\underline{|4}} \cdot \frac{\underline{|36}}{\underline{|52}}$.

6. $\frac{1}{35}$. 7. $\frac{1}{10}$. 8. $\left\{\frac{\underline{|12}\,\underline{|39}}{\underline{|9}}\right\}^2 \cdot \frac{1}{\underline{|3}\,\underline{|30}\,\underline{|51}}$.

9. $\frac{1}{\underline{|n}}$; $\left\{\frac{1}{\underline{|n}}\right\}^2$. 10. $\frac{120}{1001}$.

13. If each is put back after being drawn, $\frac{1}{134}$. If they are not put back, $\frac{\underline{|4}\,\underline{|13}}{\underline{|52}}$.

14. $\frac{\underline{|m}\,\underline{|n}\,\underline{|p+q}\,\underline{|m+n-p-q}}{\underline{|p}\,\underline{|q}\,\underline{|m-p}\,\underline{|n-q}\,\underline{|m+n}}$. 15. $1-\frac{11}{6}\left.\overline{\frac{5}{6}}\right|^4$.

16. When each is put back, $\frac{5}{144}$; $\frac{1}{144}$; $\frac{5^3}{3.12^4}$. When they are not put back, $\frac{1}{22}$; $\frac{1}{660}$; $\frac{5\,\underline{|6}\,\underline{|6}}{\underline{|11}}$.

17. $\frac{13.19.37}{17.25.49}$; 13. $\frac{\underline{|4}\,\underline{|48}\,\underline{|39}}{\underline{|52}\,\underline{|36}}$. 18. $\left\{\frac{\underline{|5}\,\underline{|95}}{\underline{|2}}\right\}^2 \cdot \frac{13}{\underline{|3}\,\underline{|2}\,\underline{|93}}$.

19. $\frac{13^3}{17.25.49}$. 20. $\frac{5\,\underline{|9}\,\underline{|11}\,\underline{|7}}{4\,\underline{|20}}$.

LVI. p. 213.

1. $2^n-1:1$. 2. $\frac{203}{23328}$. 3. (1), $\frac{3}{8}$; (2), $\frac{1}{8}$.

4. $\frac{5^5.77}{9^6}$. 5. $\frac{4}{9}$. 6. (1), $\frac{5^3}{2.6^5}$; (2), $\frac{46625}{46656}$; (3), $\frac{406}{46656}$.

7. $25:2$. 8. $671:625$. 9. $\frac{n}{2^n}$. 10. (1), $\frac{1}{8}$; (2), $\frac{3}{8}$.

11. (1), $\frac{5}{21}$; (2), $\frac{5}{28}$; (3), $\frac{15}{28}$. 12. $\frac{969}{9614}$. 14. $\frac{1}{2}-\frac{1}{2^n}$.

15. 29 : 3 against. 16. 3. 17. $\frac{5^2.2^{12}}{7^7}$. 18. $\frac{3375}{16384}$.

LVII. p. 217.

1. (1), $\frac{63}{100}$; (2), $\frac{1}{5}$; (3), $\frac{1071}{8300}$; (4), $\frac{17}{415}$.

2. (1), $\frac{6889}{10000}$; (2), $\frac{3969}{10000}$. 3. (1), $\frac{3969}{6889}$; (2), $\frac{1260}{6889}$.

4. (1), $\frac{1}{52}$; (2), $\frac{1}{52}$; (3), $\frac{1}{2704}$; (4), $\frac{1}{1352}$.

5. $\frac{1}{36}$; $\frac{1}{18}$. 6. (1), $\frac{5}{144}$; (2), $\frac{5}{24}$; (3), $\frac{25}{432}$.

LVIII. p. 223.

1. $\frac{7}{15}$. 2. $\frac{2^{m-r}}{2^m-1}$. 4. $\frac{1}{52}$. 5. (1), $\frac{1}{6}$, $\frac{5}{6^2}$, $\frac{5^2}{6^3}$, $\frac{5^3}{6^4}$.

(2), $\frac{5^4}{6^5}$, $\frac{5^5}{6^6}$, $\frac{5^6}{6^7}$, $\frac{5^7}{6^8}$; (3), $\frac{216}{671}$, $\frac{180}{671}$, $\frac{150}{671}$, $\frac{125}{671}$.

6. $\frac{1}{3}$; (1), $\frac{16}{729}$; (2), $\frac{25}{81}$. 7. $\frac{1}{1716}$. 8. $\frac{16}{21}$; $\frac{5}{42}$.

9. $\frac{1}{6}$. 10. $\frac{46656}{124393}$; $\frac{41256}{124393}$; $\frac{36481}{124393}$. 11. $\frac{11}{28}$.

LIX. p. 228.

1. $\frac{8}{11}$. 2. $\frac{(m+1)(n'+m'+1)}{(m+1)(n'+m'+1)+(m'+1)(n+m+1)}$.

3. (1), $\frac{2}{5}$; (2), $\frac{3}{5}$; (3), $\frac{8}{15}$. 4. $\frac{5}{6}$. 5. (1), $\frac{1}{42}$; (2), $\frac{5}{14}$.

6. $\frac{4}{9}$. 7. (1), $\frac{27}{41}$; (2, α), $\frac{6561}{9991}$; (2, β), $\frac{48}{73}$.

LX. p. 231.

2. $\frac{377}{550}$. 4. $\frac{11}{15}$. 5. $\frac{17}{25}$. 6. (1), $\frac{17}{45}$; (2), $\frac{8}{45}$.

7. $n^2+n-2:2$ against.

LXI. p. 233.

1. 15s. 6d. 2. $22\frac{9}{10}$s. 3. £6. 2s. 4. $29\frac{29}{40}$s.

5. $\frac{36}{91}$; $\frac{30}{91}$; $\frac{25}{91}$. 7. £1040. 8. $\frac{3}{4}$; $\frac{3}{14}$; $\frac{1}{28}$; £1. 2s.

10. $n\left(1-\frac{1}{2^r}\right)$; $\left(1-\frac{1}{2^r}\right)^n$. 12. £1666. 13s.

13. (1), *A*, 13s. 4d.; *B*, 6s. 8d. (2), *A*, 13s.; *B*, 7s.

14. £$6\frac{41}{69}$.

LXII. p. 238.

1. $\frac{2475}{2477}$. 2. $\frac{15}{43}$. 3. (1), $\frac{35}{36}$; (2), $\frac{7}{12}$; (3), $\frac{1}{36}$; (4), $\frac{1}{4}$.

4. $\frac{45}{49}$. 5. $\frac{10}{11}$.

LXIII. p. 244.

1. £$1290.\left(\frac{100}{103}\right)^{21}$ 2. $\frac{a}{(1+r)^{e+1}-r}$, where e is his expectation of life, r the interest for £1 for 1 year.

LXIV. p. 250.

1. (1), -1; (2), -2; (3), -6; (4) 3; (5), 4; (6), 39.
8. -60; -28. 10. (1), 48; (2), 44; (3), -166; (4), 24.

LXVI. p. 262.

1. 42; 138; 15; 62. 2. 0. 3. -72. 4. $l^2(u'^2-vw)+m^2(v'^2-wu)+n^2(w'^2-uv)+2mn(uu'-v'w')+$etc.
5. $2a^3b^3c^3$. 10. 0. 11. 0.

LXVII. p. 265.

1. (1), $\begin{vmatrix} a\alpha+b\beta, & a\gamma+b\delta \\ c\alpha+d\beta, & c\gamma+d\delta \end{vmatrix}$;

(2), $\begin{vmatrix} aa'+bb'+cc', & a\alpha+b\beta+c\gamma, & ax+by+cz \\ \alpha a'+\beta b'+\gamma c', & \alpha^2+\beta^2+\gamma^2, & \alpha x+\beta y+\gamma z \\ xa'+yb'+zc', & x\alpha+y\beta+z\gamma, & x^2+y^2+z^2 \end{vmatrix}$.

3. $\begin{vmatrix} x_1^2+y_1^2+z_1^2, & x_1x_2+y_1y_2+z_1z_2, & x_1x_3+y_1y_3+z_1z_3 \\ x_2x_1+y_2y_1+z_2z_1, & x_2^2+y_2^2+z_2^2, & x_2x_3+y_2y_3+z_2z_3 \\ x_3x_1+y_3y_1+z_3z_1, & x_3x_2+y_3y_2+z_3z_2, & x_3^2+y_3^2+z_3^2 \end{vmatrix}$.

4. $\begin{vmatrix} x_1, & ay_1+bz_1, & \alpha y_1+\beta z_1 \\ x_2, & ay_2+bz_2, & \alpha y_2+\beta z_2 \\ x_3, & ay_3+bz_3, & \alpha y_3+\beta z_3 \end{vmatrix}$.

LXVIII. p. 267.

1. $x=-7\frac{1}{2}$, $y=-\frac{1}{4}$, $z=5\frac{3}{4}$. 2. $x=\frac{1}{(a-b)(a-c)}$.

3. $x=18\frac{1}{2}$, $y=-10\frac{1}{2}$, $z=-10$, $u=2\frac{1}{2}$.

4. $x=-\frac{37}{47}$, $y=\frac{73}{47}$, $z=\frac{-4}{47}$, $u=\frac{41}{47}$.

5. $l^2(e^2-bc)+$ etc. $+2mn(ac-fh)+$ etc. $=0$.

LXIX. p. 270.

1. $\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}+\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right\}x^n$. 2. $(1+2^n+3^n)x^n$.

3. $\frac{7.3^n-(-1)^n}{2}x^n$. 4. $(3+2n)3^nx^n$. 5. $2.3^{n-1}(3-n)x^n$.

6. $\left\{\frac{10}{9}2^n-\frac{n}{6}2^n-\frac{(-1)^n}{9}\right\}x^n$.

8. $\frac{1}{2}(-1)^x+\frac{1}{4}(1+5\sqrt{-1})(\sqrt{-1})^x$

$$+\frac{1}{4}(1-5\sqrt{-1})(-\sqrt{-1})^x.$$

9. $\frac{\sqrt{5}+1}{2}$.

LXX. p. 272.

1. $\frac{2}{\sqrt{5}}\frac{(\sqrt{5}+1)^n-(1-\sqrt{5})^n}{(\sqrt{5}+1)^n+(1-\sqrt{5})^n}$.

2. (1), $\frac{4^{n+1}-4}{4^{n+1}-1}$; (2), $\frac{5^{n+1}+5(-1)^{n+1}}{5^{n+1}+(-1)^{n+1}}$;

(3), $\sqrt{5}\frac{(5+\sqrt{5})^n-(5-\sqrt{5})^n}{(5+\sqrt{5})^{n+1}-(3+\sqrt{5})(5-\sqrt{5})^n}$.

LXXI. p. 276.

1. $$\frac{\frac{3+\sqrt{3}}{2}(2+\sqrt{3})^{\frac{n-1}{2}}+\frac{3-\sqrt{3}}{2}(2-\sqrt{3})^{\frac{n-1}{2}}}{(2+\sqrt{3})^{\frac{n-1}{2}}+(2-\sqrt{3})^{\frac{n-1}{2}}}, \text{ when } n \text{ is odd;}$$

$$\frac{\sqrt{3}}{2}\,\frac{(\sqrt{3}+1)^2(2+\sqrt{3})^{\frac{n}{2}-1}-(\sqrt{3}-1)^2(2-\sqrt{3})^{\frac{n}{2}-1}}{(\sqrt{3}+1)(2+\sqrt{3})^{\frac{n}{2}-1}-(\sqrt{3}-1)(2-\sqrt{3})^{\frac{n}{2}-1}},$$

when n is even.

2. $\dfrac{3.4^{\frac{n-1}{2}}+2(-1)^{\frac{n-1}{2}}}{4^{\frac{n-1}{2}}+(-1)^{\frac{n-1}{2}}}$, n odd; $\dfrac{3.2^{n-1}+\frac{8}{5}(-1)^{\frac{n}{2}}}{2^{n-1}+3(-1)^{\frac{n}{2}}}$, n even.

3. $\dfrac{6^{\frac{n+1}{2}}+4}{6^{\frac{n+1}{2}}-1}$, n odd; $4\,\dfrac{6^{\frac{n}{2}}-1}{4.6^{\frac{n}{2}}+1}$, n even.

4. $\dfrac{15n-6}{3n-1}$, n odd; $\dfrac{3(12-5n)}{7-3n}$, n even.

5. $\dfrac{3-n}{1+n}$, n odd; $\dfrac{-n}{4+n}$, n even.

6. $\dfrac{a^2}{\sqrt{4a^2+9}}\times$

$$\frac{(5+\sqrt{4a^2+9})(2a^2-3+\sqrt{4a^2+9})^{\frac{n-1}{2}}+(\sqrt{4a^2+9}-5)(2a^2-3-\sqrt{4a^2+9})^{\frac{n-1}{2}}}{(2a^2-3+\sqrt{4a^2+9})^{\frac{n-1}{2}}+(2a^2-3-\sqrt{4a^2+9})^{\frac{n-1}{2}}},$$

when n is odd;

$\dfrac{2a^2(2a^2-3+\sqrt{4a^2+9})}{\sqrt{4a^2+9}}\times$

$$\frac{(2a^2-3+\sqrt{4a^2+9})^{\frac{n}{2}-1}-(2a^2-3-\sqrt{4a^2+9})^{\frac{n}{2}-1}}{(-3+\sqrt{4a^2+9})(2a^2-3+\sqrt{4a^2+9})^{\frac{n}{2}-1}-(-3-\sqrt{4a^2+9})(2a^2-3-\sqrt{4a^2+9})^{\frac{n}{2}-1}},$$

when n is even.

LXXII. p. 281.

1. $\dfrac{1-2\log 2}{\log 2}$.

7. $\dfrac{1}{2}\left\{1-\dfrac{1}{1.3.5\ldots(2n+1)}\right\}$.

12. $\dfrac{a+b}{2}$.

MISCELLANEOUS EXAMPLES.

4. (1), $-2\pm2\sqrt{2}$;

(2), $\dfrac{\sqrt{2}\pm\sqrt{2\sqrt{2}-2}}{2}$, or $\dfrac{-\sqrt{2}\pm\sqrt{-2\sqrt{2}-2}}{2}$;

(3), $y=0$, $x=0$, or $-\dfrac{1}{3}$; $y=4$, $x=9$, or $6\dfrac{2}{3}$.

5. $\dfrac{a^2}{l}+\dfrac{b^2}{m}+\dfrac{c^2}{n}=0$.

6. (1), $\dfrac{1}{40}-\dfrac{1}{8(4n+1)(4n+5)}$; (2), $\dfrac{1}{4}+\left(4n-\dfrac{5}{4}\right)5^{n-1}$.

17. (α), 1, or -2; (β), -3.

25. (1), Div.; (2), Conv.

26. (i.) $\pm\dfrac{1}{2}\sqrt{2\pm2\sqrt{2}}$; (ii.) $xy=\dfrac{1+\sqrt{m-1}\sqrt{n+1}}{1-\sqrt{m-1}\sqrt{n+1}}(=c)$,

$bx-ay=\sqrt{m-1}\,(c+ab)$.

28. $\dfrac{e^x+e^{-x}+e^{x\sqrt{-1}}+e^{-x\sqrt{-1}}}{4}$; $\dfrac{e^x-e^{-x}-\sqrt{-1}e^{x\sqrt{-1}}+e^{-x\sqrt{-1}}}{4}$;

$\dfrac{e^x+e^{-x}-e^{x\sqrt{-1}}-e^{-x\sqrt{-1}}}{4}$; $\dfrac{e^x-e^{-x}+\sqrt{-1}e^{x\sqrt{-1}}-e^{-x\sqrt{-1}}}{4}$.

31. (α), $x=3$, $y=2$, $z=1$; (β), $x=2$, $y=8$, or $x=8$, $y=2$.

34. (1), 0, $-p$, or $-\frac{p}{2} \pm \sqrt{4c^2+4cp+\frac{p^2}{4}}$.

(2), $x^2-y^2=5$, $bxy+(a+c)\,y^2=2b$.

36. 1, $-\frac{53+20\sqrt{6}}{19+8\sqrt{6}}$;

$$2x\,(m_1n_2-n_1m_2)=(mn_2-nn_1)\pm\sqrt{(m_1n_2-n_1m_2)^2+(mn_2-nn_1)^2}$$
$$2y\,(n_1m_2-m_1n_2)=(mm_2-nm_1)\pm\sqrt{(n_1m_2-m_1n_2)^2+(mm_2-nm_2)^2}.$$

38. $x=\pm\frac{a}{\sqrt{3}}$, $y=(\pm1\pm\sqrt{2})\frac{a}{\sqrt{3}}$, $z=a-(\pm2\pm\sqrt{2})\frac{a}{\sqrt{3}}$.

41. $1, 2, -1$; or -3, $\frac{-1\pm\sqrt{-23}}{4}$. 43. 20. 44. a.

45. $x=\pm a$;

$x=0$, $\frac{\sqrt{-1}\pm\sqrt[4]{-1}}{1+\sqrt{-1}}$, $\mp\sqrt{\frac{1-\sqrt{-1}}{2}}$, 1, or $\sqrt{-1}$.

$y=0$, $\frac{\sqrt{-1}\pm\sqrt[4]{-1}}{1+\sqrt{-1}}$, $\pm\sqrt{\frac{\sqrt{-1}-1}{2}}$, 0, or $\sqrt{-1}+1$.

46. $x=\frac{4a+b+c}{3}$, $y=\frac{a+4b+c}{3}$, $z=\frac{a+b+4c}{3}$.

47. x and $y=5\frac{1}{3}$, or $1\frac{1}{2}$, $z=2$.

48. $\frac{n\,(n+1)\,(3n^2+11n+28)}{12}$.

50. $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\left(\frac{a'^2}{a}+\frac{b'^2}{b}+\frac{c'^2}{c}\right)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)$.

52. $a_r=a+r\frac{b-a}{n+1}-\frac{rd}{2}(n-r+1)$; $d>0$ and $<2\frac{b-a}{n\,(n+1)}$.

53. 1. 54. $x=8$ or -3, $y=5$; $x=-8$, or 3, $y=-5$.

58. £1080. 9s.

62. (1), $x=\dfrac{c\pm\sqrt{6+2c}}{2}$, where $c=\pm\sqrt{2}$; (2), $\begin{matrix} x=a,\ b; \\ y=b,\ a. \end{matrix}$

68. (1), $\dfrac{\log 4}{\log a}$; (2), $\log_{10} 2$; (3), $x=\left(\dfrac{p}{q}\right)^{\frac{q}{p-q}}$, $y=\left(\dfrac{p}{q}\right)^{\frac{p}{p-q}}$.

70. 4, or $4+\dfrac{4}{\log 2}+\dfrac{1}{(\log 2)^2}$; $x=0, y=0$, or $x=\dfrac{\log 3}{\log 3-\log 2}$,

$$y=\frac{\log 2}{\log 3-\log 2}.$$

71. $5\pm\sqrt{11}$, 7 or 3.

76. x and y are the roots of the quadratic

$$(b^2-ac)z^2-(bc-ad)z+c^2-bd=0,$$

and $u(y-x)=b-ay,\ \ v(x-y)=b-ax.$

77. 4, 1, -2.

79. In the 66th year.

80. $\dfrac{28-4x}{(1-x)^3}+6\dfrac{x^2-x^n}{(1-x)^4}$

$$-\frac{(n^3+12n^2+39n+22)x^n-(2n^3+21n^2+51n+10)x^{n+1}+(n^3+9n^2+18n)x^{n+2}}{(1-x)^3};$$

$$\frac{x}{1-2x-3x^2}.$$

82. $\dfrac{3}{4}+$ etc. $+3\dfrac{r+1}{2^{r+2}}x^r+$ etc.

83. $-a-b-c,\quad -a^2-b^2-c^2-bc-ca-ab,$

$-a^3-b^3-c^3-a^2b-a^2c-b^2c-b^2a-c^2a-c^2b-abc,$

$-(c^{n-2}+ac^{n-3}+a^2c^{n-4}+\text{ etc. }+a^{n-2})$

$-b(c^{n-3}+ac^{n-4}+\text{ etc. }+a^{n-3})$

$-b^2(c^{n-4}+ac^{n-5}+\text{ etc. }+a^{n-4})-\text{ etc. }-b^{n-2}.$

85. (1), $\dfrac{1}{n^{n-1}}$; (2), $\dfrac{\lfloor n}{n^n}$.

86. 294, 375, 456, 537, 618.

87. $s-q+2p-3$.

89. 136; £21. 5s.

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100. $1+x^2+x^4+x^6+$ etc.

103. $x = \left\{ \dfrac{(c^6-a^3b^3)(a^3+c^3)}{(b^3+c^3)^2} \right\}^{\frac{1}{3}}$; $z = c^3 \left\{ \dfrac{(b^3+c^3)(a^3+a^3)}{(a^3b^3-c^6)^2} \right\}^{\frac{1}{3}}$.

109. ± 8.

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113. Convergent if $x<e^{-1}$, otherwise divergent.

114. $\dfrac{1}{n}\left\{1+\dfrac{1}{2}+ \;.\;.\;.\; +\dfrac{1}{n}\right\}$.

115. The roots of the equations

$$y^2-ay+\frac{b-a^2\pm\sqrt{(a^2-b^2)^2-4f}}{2};\quad a^3-ab+c=0.$$

116. $\dfrac{1}{\sqrt{4x-1}}\left\{\left(\dfrac{1+\sqrt{4x-1}}{2}\right)^{n+2}-\left(\dfrac{1-\sqrt{4x-1}}{2}\right)^{n+2}\right\}$.

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master the territorial element, receiving, however, in the course of the struggle some moderating and tempering influences from the opponent principle.

CHAPTER II.

THE PEOPLE.

1.[1] **Classes of the People.**—The English settlers in Britain were from the first divided into the two great hereditary classes of Eorls (the *principes* of Tacitus) and Ceorls,[2] both free, but the former of noble, the latter of ignoble birth. The oath of an eorl availed against that of six ceorls, and there was a corresponding difference in the amount of the weregild or compensation-money to be paid for the murder of a member of the two classes; which in the case of a ceorl was only 200 shillings (whence he was called a *twyhyndman*), but in that of an eorl 1200 shillings. Besides these distinctions between the two classes, another was introduced, which had not existed when the people dwelt in the forests of Germany. Their private wealth had then consisted of household furniture, armour, and cattle, while their land was regarded as the common property of the tribe. But after settling upon the conquered soil of Britain, they made continually increasing encroachments on the folc-land, or land common to the whole people, by converting portion after portion of it into boc-land—land held by private individuals, by book or charter. Landed wealth was at first the accompaniment of noble birth or personal merit, and when it became dissociated from these, it was gradually looked

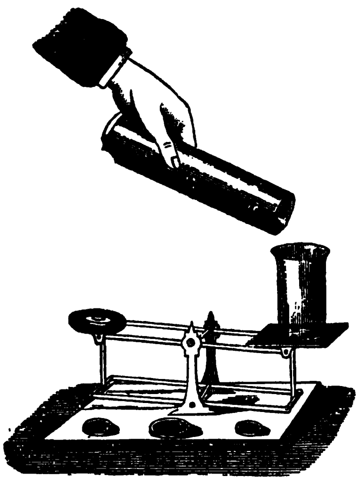
[1] For the periods of our history to which the sections marked 1–6 in the different chapters correspond, see the Preface.

[2] The words have now under the modernised forms of *earl* and *churl* acquired totally different meanings.

CHEMISTRY

In Fig. 16 is represented a very pretty experiment, showing that this gas is heavier than air. First, balance a jar with a weight. I say *balance* a jar. Is that exactly correct? Is there not something in the jar? "No," you will perhaps say, "it is empty." But think a moment. That jar is full of something, and that something has weight. It is full of air. We have balanced, then, a jar full of air. Now if, as represented, carbonic acid gas be poured into the jar on the scales, the jar will descend and the weight will rise. Why? Because there is now a gas in the jar that is heavier than air.

Fig. 16.

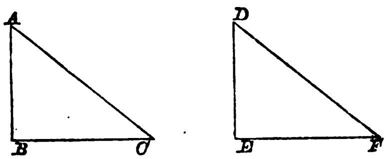


If you have a jar filled with this gas, you can take it out with a little bucket, as seen in Fig. 17. As you take one bucketful after another out, it can be poured away as water; and air will take the place of the gas as fast as it is removed.

If a soap-bubble fall into a jar of carbonic acid gas, it will not go to the bottom as it would if the jar were full of air. It will descend a little into the jar, and then ascend and remain in its open mouth. Why is this? The air that is blown into the bubble is lighter than the gas in the jar,

PROPOSITION B. THEOREM.

If two triangles have two angles of the one equal to two angles of the other, each to each, and the sides adjacent to the equal angles in each also equal; then must the triangles be equal in all respects.



In $\triangle$s ABC, DEF,

let $\angle ABC = \angle DEF$, and $\angle ACB = \angle DFE$, and $BC = EF$.

Then must $AB = DE$, *and* $AC = DF$, *and* $\angle BAC = \angle EDF$.

For if $\triangle DEF$ be applied to $\triangle ABC$, so that E coincides with B, and EF falls on BC;

then $\because EF = BC$, $\therefore F$ will coincide with C;

and $\because \angle DEF = \angle ABC$, $\therefore ED$ will fall on BA;

$\therefore D$ will fall on BA or BA produced.

Again, $\because \angle DFE = \angle ACB$, $\therefore FD$ will fall on CA;

$\therefore D$ will fall on CA or CA produced.

$\therefore D$ must coincide with A, the only pt. common to BA and CA.

$\therefore DE$ will coincide with and $\therefore$ is equal to AB,

and DF .... AC,

and $\angle EDF$ .... $\angle BAC$;

and $\therefore$ the triangles are equal in all respects. Q. E. D.

COR. Hence, by a process like that in Prop. A, we can prove the following theorem:

If two angles of a triangle be equal, the sides which subtend them are also equal. (Eucl. I. 6.)

[*Elements of Geometry. See page* 8.]

thus: if the articles had cost £1 each, the total cost would have been £2478;

∴ as they cost $\frac{1}{6}$ of £1 each, the cost will be £$\frac{2478}{6}$, or £413.

The process may be written thus:

3*s*. 4*d*. is $\frac{1}{6}$ of £1.	£2478 = cost of the articles at £1 each.
	£413 = cost .... at 3*s*. 4*d*. ...

Ex. (2). Find the cost of 2897 articles at £2. 12*s*. 9*d*. each.

£2 is 2 × £1	2897 . 0 . 0 = cost at £1 each.
10*s*. is $\frac{1}{2}$ of £1	5794 . 0 . 0 = ... £2 ....
2*s*. is $\frac{1}{5}$ of 10*s*.	1448 . 10 . 0 = ... 10*s*.....
8*d*. is $\frac{1}{3}$ of 2*s*.	289 . 14 . 0 = ... 2*s*. ....
1*d*. is $\frac{1}{8}$ of 8*d*.	96 . 11 . 4 = ... 8*d*. ....
	12 . 1 . 5 = ... 1*d*. ....
	£7640 . 16 . 9 = ... £2. 12*s*. 9*d*. each.

NOTE.—A shorter method would be to take the parts thus:

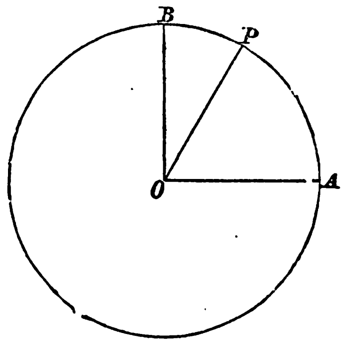
10*s*. = $\frac{1}{2}$ of £1; 2*s*. 6*d*. = $\frac{1}{4}$ of 10*s*.; 3*d*. = $\frac{1}{10}$ of 2*s*. 6*d*.

Ex. (3). Find the cost of 425 articles at £2. 18*s*. 4*d*. each.

Since £2. 18*s*. 4*d*. is the difference between £3 and 1*s*. 8*d*. (which is $\frac{1}{12}$ of £1), the shortest course is to find the cost at £3 each, and to *subtract from it* the cost at 1*s*. 8*d*. each, thus:

	£ *s*. *d*.
£3 is 3 × £1	425 . 0 . 0 = cost at £1 each.
1*s*. 8*d*. is $\frac{1}{12}$ of £1	1275 . 0 . 0 = ... £3 ....
	35 . 8 . 4 = ... 1*s*. 8*d*. each.
	£1239 . 11 . 8 = ... £2. 18*s*. 4*d*. each.

28. ***To shew that the angle subtended at the centre of a circle by an arc equal to the radius of the circle is the same for all circles.***



Let O be the centre of a circle, whose radius is r;

AB the arc of a quadrant, and therefore AOB a right angle;

AP an arc equal to the radius AO.

Then, $AP=r$ and $AB=\frac{\pi r}{2}$. (Art. 14.)

Now, by Euc. VI. 33,

$$\frac{\text{angle } AOP}{\text{angle } AOB}=\frac{\text{arc } AP}{\text{arc } AB},$$

or,

$$\frac{\text{angle } AOP}{\text{a right angle}}=\frac{r}{\frac{\pi r}{2}}$$

$$=\frac{2r}{\pi r}$$

$$=\frac{2}{\pi}.$$

Hence

$$\text{angle } AOP=\frac{2 \text{ right angles}}{\pi}.$$

Thus the magnitude of the angle AOP is independent of r and is therefore the same for all circles.

[*Trigonometry. See page* 9.]

89. Case II. **The next case in point of simplicity is that in which four terms can be so arranged, that the first two have a common factor and the last two have a common factor.**

Thus

$$\begin{aligned} x^2+ax+bx+ab &= (x^2+ax)+(bx+ab) \\ &= x(x+a)+b(x+a) \\ &= (x+b)(x+a). \end{aligned}$$

Again

$$\begin{aligned} ac-ad-bc+bd &= (ac-ad)-(bc-bd) \\ &= a(c-d)-b(c-d) \\ &= (a-b)(c-d). \end{aligned}$$

Examples.—XVIII.

Resolve into factors:

1. $x^2-ax-bx+ab$.
2. $ab+ax-bx-x^2$.
3. $bc+by-cy-y^2$.
4. $bm+mn+ab+an$.
5. $abx^2-axy+bxy-y^2$.
6. $abx-aby+cdx-cdy$.
7. $cdx^2+dmxy-cnxy-mny^2$.
8. $abcx-b^2dx-acdy+bd^2y$.

90. Before reading the Articles that follow the student is advised to turn back to Art. 56, and to observe the manner in which the operation of multiplying a binomial by a binomial produces a *trinomial* in the Examples there given. He will then be prepared to expect that in certain cases a *trinomial can be resolved into two binomial factors*, examples of which we shall now give.

91. Case III. To find the factors of

$$x^2+7x+12.$$

Our object is to find two numbers whose product is 12,
and whose sum is 7.

These will evidently be 4 and 3,

$$\therefore x^2+7x+12=(x+4)(x+3).$$

Again, to find the factors of

$$x^2+5bx+6b^2.$$

Our object is to find two numbers whose product is $6b^2$,
and whose sum is $5b$.

These will clearly be $3b$ and $2b$,

$$\therefore x^2+5bx+6b^2=(x+3b)(x+2b).$$

πρὸς ἑωυτοῦ τὸν χρησμὸν εἶναι, ἐστρατεύετο ἐς τὴν Περσέων
μοῖραν. Ὡς δὲ ἀπίκετο ἐπὶ τὸν Ἅλυν ποταμὸν ὁ Κροῖσος, 3
τὸ ἐνθεῦτεν, ὡς μὲν ἐγὼ λέγω, κατὰ τὰς ἐούσας γεφύρας διε-
βίβασε τὸν στρατόν· ὡς δὲ ὁ πολλὸς λόγος Ἑλλήνων, Θαλῆς
οἱ ὁ Μιλήσιος διεβίβασε. ἀπορέοντος γὰρ Κροίσου ὅκως οἱ 4
διαβήσεται τὸν ποταμὸν ὁ στρατὸς (οὐ γὰρ δὴ εἶναί κω
τοῦτον τὸν χρόνον τὰς γεφύρας ταύτας), λέγεται παρεόντα
τὸν Θαλῆν ἐν τῷ στρατοπέδῳ ποιῆσαι αὐτῷ τὸν ποταμόν,
ἐξ ἀριστερῆς χειρὸς ῥέοντα τοῦ στρατοῦ, καὶ ἐκ δεξιῆς ῥέειν·
ποιῆσαι δὲ ὧδε. ἄνωθεν τοῦ στρατοπέδου ἀρξάμενον, διώ- 5
ρυχα βαθέην ὀρύσσειν, ἄγοντα μηνοειδέα, ὅκως ἂν τὸ
στρατόπεδον ἱδρυμένον κατὰ νώτου λάβοι, ταύτῃ κατὰ τὴν
διώρυχα ἐκτραπόμενος ἐκ τῶν ἀρχαίων ῥεέθρων, καὶ αὖτις
παραμειβόμενος τὸ στρατόπεδον, ἐς τὰ ἀρχαῖα ἐσβάλλοι·
ὥστε, ἐπεί τε καὶ ἐσχίσθη τάχιστα ὁ ποταμός, ἀμφοτέρῃ 6
διαβατὸς ἐγένετο. οἱ δὲ καὶ τὸ παράπαν λέγουσι καὶ τὸ
ἀρχαῖον ῥέεθρον ἀποξηρανθῆναι. ἀλλὰ τοῦτο μὲν οὐ προσ-

§ 2. **πρὸς ἑωυτοῦ**] *E sua parte.* *πρός* = from the direction of (110. 2, n.), from the point of view of, and so favourable towards. Cf. *πρὸς τῶν ἐχόντων, Φοῖβε, τὸν νόμον τίθης,* Eur. *Alc.* 57.

§ 3. **τὰς ἐούσας γ.**] The plural of a single bridge (205. 3, n.).

§ 4. **ταύτας** = *τὰς ἐούσας*, above. **λέγεται**] Hdt.'s doubts about this story are prob. due to chronological difficulties (Ab.). 'The exact year of Thales' birth and the date of his death cannot be known.' Clinton.

ἐξ ἀριστερῆς] This implies that the army was marching, or that the camp was facing, upstream (i. e. southwards) at the time.

καὶ ἐκ δεξ.] 'Partly on the right hand as well' (§ 6).

§ 5. **ὅκως ἂν...λάβοι**] A common construction in Hdt., as in Homer. Cf. 91. 2; 99. 3; 152. 2. Thuc. has *μὴ ἂν—ἐπιπλεύσειαν*, II. 93. 2. Prob. *ἂν* renders the object in view rather less definite than it would otherwise be, by implying the existence of some condition := 'if possible.' 'With the opt. *ὡς ἄν, ὅπως ἄν* = *quomodo* or *ut*. *προμηθοῦνται ὅπως ἂν εὐδαιμονοίης* is derived from the direct interrogative, *πῶς ἂν (εἰ δυνατὸν εἴη) εὐδαιμονοίης;*' Madv. *G. S. App.* 302. Tr. 'that so peradventure (the river) might take the camp, there pitched, in the rear (i. e. might flow on the western side of the camp), having on this side been diverted from its ancient course into the channel.'

§ 6. **καὶ ἐσχίσθη**] '*καί* leads one to expect a second *καί* before *διαβατός* which is omitted.' Kr. More prob. *καί* = 'actually,' the mere purpose (*ὅκως* above) now having the performance superadded.

καὶ τὸ παράπαν] 117. 1, n.

καὶ τὸ ἀρχ.] *καί* belongs to the object of *λέγ.* = 'say this also, viz. that.'

διέβησαν] 'How did they cross (on this supposition)?' i.e. how could they have crossed? Cf. 187. 5, n. Hdt.'s objection is hardly a valid one, since they might have dammed up the new stream and again diverted the river (into its old bed).

ΗΛ. [*interrupting*] τί τῶν ἀπόντων ἢ τί τῶν ὄντων πέρι;
ΠΡ. [*solemnly*] λαβεῖν φίλον θησαυρόν, ὃν φαίνει θεός.
ΗΛ. ἰδού, καλῶ θεούς.
[*clasping her hands*] ἢ τί δὴ λέγεις, γέρον;
ΠΡ. βλέψον νυν ἐς τόνδ', ὦ τέκνον, τὸν φίλτατον.
[*turning her round to* Orestes.]
ΗΛ. [*sadly*] πάλαι δέδοικα, μὴ σύ γ' οὐκέτ' εὖ φρονῇς.
ΠΡ. οὐκ εὖ φρονῶ 'γὼ σὸν κασίγνητον βλέπων;
ΗΛ. [*starting suddenly*]
πῶς εἶπας, ὦ γεραί', ἀνέλπιστον λόγον;
ΠΡ. [*emphatically*] ὁρᾶν 'Ορέστην τόνδε τὸν 'Αγαμέμνονος.
ΗΛ. ποῖον χαρακτῆρ' εἰσιδών, ᾧ πείσομαι; [*incredulous*]
ΠΡ. [*pointing at a scar in* Orestes' *forehead*]
οὐλὴν παρ' ὀφρύν, ἥν ποτ' ἐν πατρός δόμοις
νεβρὸν διώκων σοῦ μέθ' ἠμάχθη πεσών.
ΗΛ. πῶς φῄς; ὁρῶ μὲν πτώματος τεκμήριον.
[*astounded, but still hesitating.*]
ΠΡ. ἔπειτα μέλλεις προσπίτνειν τοῖς φιλτάτοις;
ΗΛ. [*resolved*] ἀλλ' οὐκέτ', ὦ γεραιέ· συμβόλοισι γὰρ
τοῖς σοῖς πέπεισμαι θυμόν. [*she rushes in a transport of joy into her brother's arms.*] ὦ χρόνῳ φανείς,
ἔχω σ' ἀέλπτως. ΟΡ. κἀξ ἐμοῦ γ' ἔχει χρόνῳ.
ΗΛ. οὐδέποτε δόξασ'. ΟΡ. οὐδ' ἐγὼ γὰρ ἤλπισα.
ΠΡ. ἐκεῖνος εἶ σύ;
ΟΡ. σύμμαχός γέ σοι μόνος,
ἢν ἐκσπάσωμαί γ' ὃν μετέρχομαι βόλον.
πέποιθα δ'. ἢ χρὴ μηκέθ' ἡγεῖσθαι θεούς,
εἰ τἄδικ' ἔσται τῆς δίκης ὑπέρτερα. [*with confidence.*]

378 *But loose in morals.* Such a one as George Selwyn's chaplain and parasite, Dr Warner. "In letter after letter he (Dr Warner) adds fresh strokes to the portrait of himself, not a little curious to look at now that the man has passed away; all the foul pleasures and gambols in which he revelled, played out; all the rouged faces into which he leered, worms or skulls; all the fine gentlemen whose shoebuckles he kissed, laid in their coffins."—THACKERAY's *George III.* See also Goldsmith's *Citizen of the World*, No. 58, "A Visitation Dinner;" Knight's *History of England*, vol. vii., p. 109.

384 *Scrawls a card.* Writes his name on a visiting card. Visiting cards in the last century were not the plain bits of pasteboard which we see now-a-days, they had generally some vignette or ingenious device engraved on them. Specimens may be seen at Dresden which Raphael Mengs drew and Raphael Morghen engraved.

385 *Rout.* A crowd or crush, the fashionable term in the last century for what is now called an "at home." For an amusing account of a rout to which Porson was inveigled, see Landor's *Imaginary Conversations*, Southey and Porson.

"*Southey*—Why do you repeat the word *rout* so often?

"*Porson*—Not because the expression is new and barbarous, I do assure you, nor because the thing itself is equally the bane of domestic and polite society."

389 *By infidelity.* "This worthy clergyman takes care to tell us that he does not believe in his religion."—THACKERAY, *loc. cit.*

390 *A sinecure.* Especially applied to a benefice without the cure of souls.

397–408. A free paraphrase and amplification of 1 Tim. iii. 1–11, and Titus i. 7–9.

409 *Rostrum.* More correctly "rostra," the stage or pulpit for speakers in the Roman forum, so called from being ornamented with the beaks of ships taken from the Antians, A.U.C. 416.

410–414 See remarks on Cowper's wit and humour, in Introduction.

420 *Conceit of.* Vanity on account of.

423 *Tropes.* Trope, Greek τρόπος, properly a word turned from its natural sense, then applied more generally to any rhetorical ornament.

430 *Avaunt.* French "avant," Latin "ab ante," move on, begone.

431 *Theatric*; *-ic* is from the French *-ique.* The additional adjectival termination *-al* in the modern theatrical arose from the adjectives in *-ic* (logic, mathematics, or more correctly mathematic, domestic, &c.) acquiring the force of substantives.

435 *Curious.* Inquisitive.

436 *Nasal twang.* A relic of Puritanism, and generally supposed,

their kind, and of every creeping thing of the earth after his kind." Sufficient food was also to be provided: "take thou unto thee of all food that is eaten, and thou shalt gather it to thee, and it shall be for food for thee and for them" [GEN. vi. 19-21].

To make all these preparations required a strong belief in God on the part of Noah. The world around him utterly disbelieved the message which he conveyed to it during many years of preparation as the "preacher of righteousness" [2 PET. ii. 5], while God's longsuffering waited [1 PET. iii. 20]. Our Lord says that "they were eating and drinking, marrying and giving in marriage, until the day that Noah entered into the ark, and knew not until the flood came and took them all away" [MATT. xxiv. 38; LUKE xvii. 26]. But though all the world disregarded, Noah was entitled to be enrolled among the number of St. Paul's "elders who obtained a good report," for his faith made him believe in the things of which God gave him warning "though not seen as yet" [HEB. xi. 7], and it is recorded of him, "Thus did Noah; according to all that God commanded him so did he" [GEN. vi. 22].

The Ark which Noah built in obedience to the Divine command was not a navigable ship, but a great wooden "coffer," or water-tight chest, made so as to float about steadily upon the water.[1]

It was built of cypress or "gopher" wood, and covered with pitch within and without to secure it against leakage from the flood below or the rain above. The size of the ark is distinctly given as being 300 cubits in length by 50 cubits in width, and 30 cubits in height. The cubit is reckoned at about 21 inches, and we are thus able to compare the size of the ark with that of our large iron and wooden ships of modern days.[2]

	Length.	Breadth.	Depth.
The Ark . .	525 feet	87 feet 6 inches	52 feet 6 inches
Duke of Wellington	240 feet	60 feet	72 feet 4 inches
Great Eastern .	680 feet	83 feet	58 feet

[1] Its object being the same as that of the "ark" in which the infant Moses was placed when cast into the Nile in obedience to the edict of Pharaoh.

[2] The proportions of the ark are exactly those of the human body, viz., 10'+1'6+1'; and the capacity of these proportions for stowage has been proved by experiments in Holland and Denmark to be a third greater than that of vessels as built for ordinary sailing purposes. That of the Ark was thus about the same as that of the Great Eastern.

Twenty-ninth Lesson.

CHANTING.

CHANTING is the arrangement of prose in a rhythmical form. The psalms, canticles, &c. are sung or chanted to melodies called CHANTS, which are either SINGLE or DOUBLE.

The melody of a single chant is, for convenience, written in phrases of seven bars of two minims each or their value.

The first half of a chant has three, the second four bars.

The first half is called the *mediation*, the second the *cadence.*

TURNER.

First half. *Second half.*

A double chant is simply a single chant form repeated.

ATTWOOD.

First half a. *Second half a.* *First half b.* *Second half b.*

A single chant is arranged to fit one verse of the psalms, a double chant two; for the long psalms quadruple chants, of which the phrase or melody is designed to include four verses, have been written.

A changeable chant is one whose key-chord may be either

[*The Chorister's Guide. See page* 28.]

(especially in winter), and only a limited number of troops can march along one road. Thus all roads leading out of a fortress are to some extent like causeways across a marsh, for practical purposes. The difficulty is diminished by acting at night, and by making feints.

24. Fort St. Georges was on the east, La Favorita on the north side, both on the outside of the lakes. A tête-de-chaussée is a fort which commands and "caps" a road, as a tête-de-pont does a bridge.

25. "Considered himself able to obtain."

26. Detached, that is, from the army now under the Archduke Charles. Till this new force, under a new general, should arrive, Melas was left in command of what remained of Beaulieu's army, now in retreat up the valley of the Adige. Beaulieu himself was recalled.

27. The district called the Vorarlberg lies between the Lake of Constance and the Tyrol. The Tyrolese attachment to the House of Austria is famous. In 1809, Napoleon wanted to take the Tyrol from Austria, and give it to Bavaria, setting up the latter as a rival power to Austria. The Tyrolese resisted. [Story of Hofer.]

28. [Why did not Bonaparte cross the Adige, or else ascend it, and make for the Danube?]

29. "Dependent on" (comp. the English "irrelevant") . . . "invested with," i.e. holding. These little domains were only nominally dependent on the empire; in reality they were part of the territory of Genoa, and contributed to its militia. "The empire" had only eight years more to live. When Francis II. saw that he had lost all real power as emperor, he threw it up altogether, and took the title of Emperor of Austria instead.

30. [St. Januarius.]

31. There were also six thousand English in Corsica, who might have reinforced an army attacking Bonaparte from the south. [Have English troops ever been in North Italy? Only once, I believe.]

32. In its lower course, the Po is higher than the surrounding country, thanks to the deposits brought down from the Alps, which raise its bed incessantly. It is walled in by high embankments, kept in order by a staff of engineers, as in Holland. But, in spite of their efforts, the river sometimes breaks through.

33. "Referred the question of peace to."

34. Napoleon had strange good fortune in one respect: his enemies never attacked him at the same moment. In this campaign he could hardly have resisted a flank attack from a Papal and Neapolitan army combined with that of the Austrians. So, when he beat Austria at Austerlitz, Prussia on his left flank was holding back; when he beat Prussia at Jena, Austria on his right flank was passive; when he invaded Russia, neither Prussia nor Austria stirred; when at last they did combine in one attack, they were more than a match for him, and he was ruined in the great battle of 1813.

[*Notes to Campaigns of Napoleon*. See page 27.]

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